

Friction in a Model of Hamiltonian Dynamics

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“A moving body will come to rest as soon as the force pushing it no longer acts on it in the manner necessary for its propulsion.”—— Aristotle

Abstract: We study the motion of a heavy tracer particle weakly coupled to a dense ideal Bose gas exhibiting Bose-Einstein condensation. In the so-called mean-field limit, the dynamics of this system approaches one determined by nonlinear Hamiltonian evolution equations describing a process of emission of Cerenkov radiation of sound waves into the Bose-Einstein condensate along the particle’s trajectory. The emission of Cerenkov radiation results in a friction force with memory acting on the tracer particle and causing it to decelerate until it comes to rest.

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1. Introduction

On the basis of abundant everyday experience, Aristotle formulated a naive dynamical law, which, roughly speaking, says that the velocity of a moving body is proportional to the external force acting on it and that it approaches a state of rest as soon as the force that has propelled it no longer acts on it. This law captures the phenomenon of friction. We have learnt from the discoveries of Galileo, Newton and their followers that Aristotle's law is not the right starting point for the development of analytical mechanics. It is the acceleration rather than the velocity of a body moving in empty space that is proportional to the force acting on it. Yet, particle motion with friction caused by dissipative processes is an omni-present phenomenon. It is thus of considerable interest to analyze how this type of motion emerges from the Hamiltonian dynamics of particles coupled to a dispersive environment.

The problem of constructing the effective dynamics of a particle coupled to dispersive reservoirs has previously been studied, e.g. in [1, 2, 6, 8] and references given there. In [5], we have introduced a family of quantum mechanical models describing tracer particles moving through a Bose gas exhibiting Bose Einstein condensation. We have identified a regime in which the dynamics of this system approaches one governed by classical Hamiltonian evolution equations. Similar equations have also been considered in [7]. This so-called “mean field regime” is characterized as follows:

- (i) The mass of the tracer particle is chosen to be $M_p = NM$, $M > 0$, and the potential of external forces acting on it is $V_p(X) = NV(X)$, where X is the particle position, and N is a parameter ranging over the interval $[1, \infty)$.
- (ii) The mean density of the Bose gas is chosen to be $\rho = N \frac{\rho_0}{g^2}$, $\rho_0 > 0$, where g is a real parameter. The coupling constant of two-body forces between atoms in the Bose gas is chosen to be $\lambda = \frac{\kappa}{N}$, $\kappa \geq 0$. The two-body forces are derived from a potential $\phi(x - y)$ assumed to be **spherically symmetric**, of **short range** and of **positive type**, (i.e., the two-body force is repulsive in average); x and y denote the positions of two atoms in the Bose gas. The mass of an atom is denoted by m .
- (iii) The interaction between the tracer particle and an atom in the Bose gas is described by a two-body potential $gW(x - X)$, where the coupling constant g is the same parameter as the one introduced in (ii) and W is **spherically symmetric** and of **short range**.

The **mean-field regime** corresponds to the limit

$$N \rightarrow \infty. \quad (1.1)$$

Remark 1. If the two-body Schrödinger operator $-\frac{1}{2m}\Delta_x + gW(x)$ has bound states and the coupling constant κ is strictly positive then the “effective mass” of the tracer particle

can be argued to be proportional to Nm , because it forms a bound state with $O(N)$ atoms of the Bose gas. In this situation, the assumption that $M_p = NM$, see (i), is presumably superfluous; see [5].

Heuristic arguments (see [5]), which can be made rigorous, mathematically, for ideal Bose gases ($\kappa = 0$) indicate that, in the mean-field limit (1.1), the dynamics of the system is described by the following classical, nonlinear Hamiltonian evolution equations:

$$\dot{X}_t = \frac{P_t}{M}, \quad \dot{P}_t = -\nabla_X V(X_t) + g \int \nabla_x W(X_t - x) \left(|\alpha_t(x)|^2 - \frac{\rho_0}{g^2} \right) dx, \quad (1.2)$$

$$i\dot{\alpha}_t(x) = \left(-\frac{1}{2m} \Delta + gW(X_t - x) \right) \alpha_t(x) + \kappa \phi * \left(|\alpha_t(y)|^2 - \frac{\rho_0}{g^2} \right) \alpha_t(x). \quad (1.3)$$

In Eqs. (1.2) and (1.3), $X_t \in \mathbb{R}^3$ and $P_t \in \mathbb{R}^3$ are the position and momentum of the tracer particle, respectively, at time t , and $\alpha_t(x)$ is the Ginzburg-Landau order-parameter field describing the state of the Bose gas at time t (in the mean-field limit). The interpretation of $|\alpha_t(x)|^2$ is that of the density of atoms at the point x of physical space \mathbb{R}^3 , at time t ; the global phase of α_t is not an observable quantity. The symbol $*$ in (1.3) denotes convolution.

We impose the conditions that $\nabla \alpha_t$ is square-integrable in x and that $|\alpha_t|^2 - \frac{\rho_0}{g^2}$ is integrable. This defines an affine space of complex-valued functions denoted by $\tilde{\Gamma}_{BG}$ whose tangent space can be chosen to be some weighted Sobolev space. We define Γ to be the Cartesian product of \mathbb{R}^6 (the tracer particle's phase space) with Γ_{BG} . The space Γ is the phase space of the system. It is equipped with the standard symplectic form

$$\omega = \sum_{n=1}^3 dX^n \wedge dP_n - i \int d\bar{\alpha} \wedge d\alpha. \quad (1.4)$$

Equations (1.2) and (1.3) then turn out to be the Hamiltonian equations of motion corresponding to the Hamilton functional

$$\begin{aligned} H(X, P, \alpha, \bar{\alpha}) = & \frac{P^2}{2M} + V(X) + \int dx \left\{ \frac{1}{2m} |\nabla \alpha(x)|^2 + gW(X - x) (|\alpha(x)|^2 - \frac{\rho_0}{g^2}) \right\} dx \\ & + \frac{\kappa}{2} \int \int dx dy (|\alpha(y)|^2 - \frac{\rho_0}{g^2}) \phi(y - x) (|\alpha(x)|^2 - \frac{\rho_0}{g^2}). \end{aligned} \quad (1.5)$$

This functional is sometimes called the ‘‘Gross-Pitaevskii functional.’’ It exhibits a global $U(1)$ symmetry,

$$\alpha(x) \rightarrow e^{i\theta} \alpha(x), \quad \bar{\alpha}(x) \rightarrow e^{-i\theta} \bar{\alpha}(x), \quad (1.6)$$

where θ is an x -independent phase. This symmetry is broken by a choice of boundary conditions at ∞ . In this paper, we choose zero-vorticity boundary conditions,

$$\alpha(x) \xrightarrow{|x| \rightarrow \infty} \sqrt{\frac{\rho_0}{g^2}}.$$

We introduce a ‘‘fluctuation field’’, β , by setting

$$\alpha(x) := \sqrt{\frac{\rho_0}{g^2}} + \beta(x), \quad (1.7)$$

with $\beta(x) \rightarrow 0$, as $|x| \rightarrow \infty$. The equations of motion then read

$$\dot{X}_t = \frac{P_t}{M}, \quad \dot{P}_t = -\nabla_X V(X_t) + g \int \nabla_x W(X_t - x) \left(|\beta_t(x)|^2 + 2\sqrt{\frac{\rho_0}{g^2}} \operatorname{Re} \beta_t(x) \right) dx, \quad (1.8)$$

$$i\dot{\beta}_t(x) = \left(-\frac{1}{2m} \Delta + gW(X_t - x) \right) \beta_t(x) + \sqrt{\rho_0} W(X_t - x) + \kappa \left(\phi * \left(|\beta_t|^2 + 2\sqrt{\frac{\rho_0}{g^2}} \operatorname{Re} \beta_t \right) \right)(x) \left(\beta_t(x) + \sqrt{\frac{\rho_0}{g^2}} \right). \quad (1.9)$$

The Hamilton functional giving rise to these equations is obtained from (1.5) after inserting the substitution (1.7). Equations (1.8) and (1.9) have stationary (time-independent) solutions, and if the external force vanishes ($V \equiv 0$) they have traveling wave solutions, provided the speed of the particle is smaller than or equal to the speed of sound in the condensate; see [5]. If the initial speed of the particle is larger than the speed of sound of the condensate a non-zero friction force is generated, because the particle emits sound waves into the condensate (Cerenkov radiation) and hence loses energy until its speed drops to the speed of sound, whereupon it continues to move ballistically, accompanied by a “splash” in the order-parameter field β . (Quantum mechanically, this splash corresponds to a coherent state of gas atoms and causes decoherence in particle-position space, allowing for an essentially “classical” detection of the particle trajectory.)

The following models are of interest; see [5]:

B -Model: $\kappa = 0$ (ideal Bose gas), $g \rightarrow 0$, see [4].

C -Model: $\kappa = 0$ and $g \neq 0$; see [3].

E -Model: $\kappa\rho_0/g^2 = \mu = \text{const.}$, and $g, \kappa \rightarrow 0$

G -Model: $\kappa > 0$ and $g \neq 0$.

The B -model is a special case of the E -model ($\mu = 0$). The C -model is a little harder to analyze than the B -model, and one must assume that the operator $-\frac{1}{2m} \Delta + gW$ does not have bound states or zero-energy resonances. (Bound states would cause an instability in the C -model and would presumably lead to “run-away” solutions; see [5]). Work on the E -model is in progress.

For further discussions of the physics background of these models, special solutions of the equations of motion and references to other studies of related problems we refer to [5].

In this paper, we focus our attention on the simplest model, the B -model, with $V \equiv 0$. The equations of motion are then given by

$$\dot{X}_t = \frac{P_t}{M}, \quad \dot{P}_t = 2\sqrt{\rho_0} \operatorname{Re} \langle \nabla_x W^{X_t}, \beta_t \rangle; \quad (1.10)$$

$$i\dot{\beta}_t(x) = -\frac{1}{2m} \Delta \beta_t + \sqrt{\rho_0} W^{X_t}, \quad (1.11)$$

where

$$W^X(x) := W(X - x), \quad \langle f, g \rangle := \int \bar{f}(x) g(x) dx. \quad (1.12)$$

The corresponding Hamilton functional is given by

$$H(X, P; \bar{\beta}, \beta) := \frac{|P|^2}{2M} + \frac{1}{2m} \int_{\mathbb{R}^3} |\nabla \beta|^2 dx + 2\sqrt{\rho_0} \int_{\mathbb{R}^3} W^X \operatorname{Re} \beta dx. \quad (1.13)$$

The main result established in this paper says that if the initial kinetic energies of the particle and of the fluctuations in the Bose gas, as described by β , are small enough, as compared to $\rho_0 \int dx W(x)$, (and if $\beta_{t=0}$ decays sufficiently rapidly at ∞) then the solutions of (1.10), (1.11) have the properties that

$$\begin{aligned} |P_t| &\searrow 0, \text{ integrably fast in time } t, \\ X_t &\rightarrow X_\infty, \text{ and} \\ \beta_t &\rightarrow 2m\sqrt{\rho_0}\Delta^{-1}W^{X_\infty}, \text{ as } t \rightarrow \infty. \end{aligned} \quad (1.14)$$

By scaling time t , space x and the fluctuation field β , denoting the new variables again by t , x and β , we can write the equations of motion of the B -model in the form

$$\dot{X}_t = P_t, \quad \dot{P}_t = \nu \operatorname{Re} \langle \nabla_x W^{X_t}, \beta_t \rangle; \quad (1.15)$$

$$i\dot{\beta}_t(x) = -\frac{1}{2}\Delta\beta_t + W^{X_t}, \quad (1.16)$$

with

$$|\hat{W}(0)| = (2\pi)^{-\frac{3}{2}} \left| \int d^3x W(x) \right| = 1. \quad (1.17)$$

Henceforth, we study (1.15) and (1.16), with the normalization condition (1.17) imposed.

Technically, the main difficulty surmounted in our paper is to construct solutions of a certain semi-linear integro-differential equation, (see (3.8) below) whose linearization (in P) takes the form

$$\partial_t q_t = Z \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle \int_0^t q_s ds - Z \operatorname{Re} \langle W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} q_s ds W \rangle, \quad (1.18)$$

where q_t is a component of P_t , and $Z > 0$ is a positive constant. Using properties of the solution of (1.18), we reformulate the original semi-linear integro-differential equation in such a way that a fixed-point theorem becomes applicable to construct solutions, provided that the initial conditions are small enough.

Our paper is organized as follows: in Sect. 2 we carefully state our main result, Theorem 2.1. In Sect. 3 we rewrite (1.15) and (1.16) in a more convenient form. This leads to an equation for P_t containing a linear part and a higher-order nonlinear part. The linearized equation is then carefully studied in Sect. 4, which, technically, is the core of the present paper. The proof of our main result is completed in Sect. 5. In several appendices (Appendices A through E) some auxiliary technical results are established.

2. The Main Theorem

In this section, we summarize our main results concerning the solutions of the equations of motion (1.15) and (1.16). For this purpose, we define a certain interval $I \subset (0, 1)$ of real numbers, δ , as follows:

$$I := \left\{ \delta \mid \delta > 0, \int_0^1 \frac{1}{1 + (1-r)^{\frac{1}{2}}} (1-r)^{-\frac{1}{2}} \left[\frac{1}{1-2\delta} (r^{-\frac{1}{2}} - r^{-\delta}) + r^{\frac{1}{2}-\delta} \right] dr < \pi \right\}. \quad (2.1)$$

Numerical evaluation of the integral on the right-hand side of (2.1) on a computer shows that I is non-empty, with

$$I_{sup} := \sup_{\delta} \{\delta \mid \delta \in I\} \approx 0.66.$$

The following theorem is the main result established in this paper.

Theorem 2.1. *Suppose that the two-body potential W in (1.13), (1.15) and (1.16) is smooth, spherically symmetric and of fast decay at infinity with $(2\pi)^{-\frac{3}{2}} |\int dx W(x)| = 1$, and that the constant v in (1.15) is $O(1)$. Then, given any $\delta \in I$, there exists $\epsilon_0 = \epsilon_0(\delta)$ such that if*

$$\|\langle x \rangle^4 \beta_0\|_2, \quad |P_0| \leq \epsilon_0,$$

then

$$|P_t| \leq ct^{-\frac{1}{2}-\delta} \quad \text{as } t \rightarrow \infty, \quad (2.2)$$

for some constant $c = c(\delta, \epsilon_0) < \infty$. Moreover,

$$\lim_{t \rightarrow \infty} \|\beta_t + 2(-\Delta)^{-1} W^{X_t}\|_{\infty} = 0. \quad (2.3)$$

Here the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$, are understood as follows: for any function $f : \mathbb{R}^n \rightarrow \mathbb{C}$,

$$\|f\|_p := \left[\int_{\mathbb{R}^n} |f|^p dx \right]^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and

$$\|f\|_{\infty} = \sup\{t \mid \mu[x \mid |f(x)| \geq t] > 0\},$$

where μ is the Lebesgue measure in \mathbb{R}^n .

The main theorem will be proven in Sect. 5. In Sect. 3, we derive a law/equation describing the effective dynamics of the tracer particle after eliminating the degrees of freedom of the Bose gas, and, in Sect. 4, we study the effective particle dynamics and, in particular, analyze its linearization, which represents a key ingredient of our analysis. Some heuristic arguments indicating what the true decay of the particle momentum P_t in time t might be are presented in Appendix E.2.

3. Effective Particle Dynamics, and Local Wellposedness

To begin with, we recast Eqs. (1.15) and (1.16) in a form amenable to precise mathematical techniques. We start with transforming Eq. (1.15) into a convenient form. We introduce a new field, δ_t , by

$$\beta_t =: 2\Delta^{-1} W^{X_t} + \delta_t. \quad (3.1)$$

The first term on the right-hand side of (3.1) describes a “splash” in the Bose gas (a depletion if W is repulsive, and an accretion if W is attractive), while the field δ_t describes

the emission of sound waves into the Bose gas by the tracer particle. In terms of the field δ_t the equations of motion (1.15) and (1.16) are seen to be

$$\begin{aligned}\dot{X}_t &= P_t, \quad \dot{P}_t = \nu \operatorname{Re} \langle \nabla_x W^{X_t}, \delta_t \rangle, \\ i\dot{\delta}_t(x) &= -\frac{1}{2} \Delta \delta_t + 2i \Delta^{-1} P_t \cdot \nabla_x W^{X_t},\end{aligned}\quad (3.2)$$

with

$$\delta_0 = -2\Delta^{-1} W^{X_0} + \beta_0, \quad (3.3)$$

where, in contrast to the term $-2\Delta^{-1} W^{X_0}$, β_0 has good decay at infinity. Using Duhamel's principle, we obtain for δ_t ,

$$\delta_t = -2e^{i\frac{\Delta}{2}t}(\Delta)^{-1} W^{X_0} + e^{i\frac{\Delta}{2}t} \beta_0 + 2 \int_0^t e^{i\frac{\Delta}{2}(t-s)} (\Delta)^{-1} P_s \cdot \nabla_x W^{X_s} ds. \quad (3.4)$$

Plugging this equation into the equation for P_t , we find that

$$\begin{aligned}\dot{P}_t &= \nu \operatorname{Re} \langle \nabla_x W^{X_t}, \delta_t \rangle \\ &= \nu \operatorname{Re} \langle \nabla_x W^{X_t}, e^{i\frac{\Delta}{2}t} \beta_0 \rangle \\ &\quad + 2\nu \operatorname{Re} \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} W^{X_0-X_t} \rangle \\ &\quad - 2\nu \operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} P_s \cdot \nabla_x W^{X_s-X_t} ds \rangle,\end{aligned}\quad (3.5)$$

where we have used that $\langle f^X, g^Y \rangle = \langle f, g^{Y-X} \rangle$. Next we use that

$$W^{Y-X_t} = W^{Y-X_0} - \int_0^t P_s \cdot \nabla_x W^{Y-X_s} ds,$$

to arrive at

$$\begin{aligned}&\operatorname{Re} \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} W^{X_0-X_t} \rangle \\ &= \operatorname{Re} \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} [W^{X_0-X_t} - W] \rangle \\ &= \operatorname{Re} \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} \int_0^t P_s \cdot \nabla_x W ds \rangle \\ &\quad + \operatorname{Re} \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} \int_0^t P_s \cdot \nabla_x [W^{X_0-X_s} - W] ds \rangle;\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} P_s \cdot \nabla_x W^{X_s-X_t} ds \rangle \\ &= \operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} P_s \cdot \nabla_x W ds \rangle \\ &\quad + \operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} P_s \cdot \nabla_x [W^{X_s-X_t} - W] ds \rangle.\end{aligned}$$

Plugging these identities into (3.5), we find that

$$\begin{aligned} \dot{P}_t &= F_P + 2\nu Re \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} \int_0^t P_s ds \cdot \nabla_x W \rangle \\ &\quad - 2\nu Re \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} P_s \cdot \nabla_x W \rangle ds, \end{aligned} \quad (3.6)$$

where

$$F_P := B_0 + B_1 + B_2 \quad (3.7)$$

with

$$\begin{aligned} B_0 &:= \nu Re \langle \nabla_x W^{X_t}, e^{i\frac{\Delta}{2}t} \beta_0 \rangle, \\ B_1 &:= 2\nu Re \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} \int_0^t P_s \cdot \nabla_x [W^{X_0-X_s} - W] ds \rangle \end{aligned}$$

and

$$B_2 := -2\nu Re \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} P_s \cdot \nabla_x [W^{X_s-X_t} - W] ds \rangle,$$

the time argument, t , being omitted in $B_i = B_i(t)$. Note that only B_0 depends on the initial condition, β_0 , of the Bose gas.

Equation (3.6) can be simplified somewhat as follows. Using that W is spherically symmetric, we have that

$$\begin{aligned} \langle \partial_k W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} P_s \cdot \nabla_x W \rangle &= \langle \partial_k W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} p_s^{(k)} \partial_k W \rangle \\ &= \frac{1}{3} p_s^{(k)} \langle W, e^{i\frac{\Delta}{2}t} W \rangle, \quad k = 1, 2, 3, \end{aligned}$$

where $\partial_k := \frac{\partial}{\partial x_k}$, $P_s = (p_s^{(1)}, p_s^{(2)}, p_s^{(3)}) \in \mathbb{R}^3$.

Equation (3.6) is then seen to be equivalent to the following equation (or law):

$$\dot{P}_t = L(P)(t) + F_P(t), \quad (3.8)$$

where L is a linear operator on the space of momentum trajectories $\{P_s\}_{0 \leq s < \infty}$ given by

$$L(P) := \left(L(p^{(1)}), L(p^{(2)}), L(p^{(3)}) \right)$$

with

$$L(h)(t) := \frac{2}{3} \nu Re \langle W, e^{i\frac{\Delta}{2}t} W \rangle \int_0^t h_s ds - \frac{2}{3} \nu Re \langle W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} h_s ds W \rangle, \quad (3.9)$$

for an arbitrary function $h : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$. Equation (3.8) is the law describing the effective dynamics of the tracer particle.

Next, we study the well-posedness of Eq. (3.8). Our results are summarized in the following theorem.

Theorem 3.1. *Equation (3.8) is locally well-posed: for $P_0 \in \mathbb{R}^3$ and $\langle x \rangle^4 \beta_0 \in L^2$, there exists a positive time $T(|P_0|, \|\langle x \rangle^4 \beta_0\|_2)$ such that a solution P_t of Eq. (3.8) exists for any time t , with $0 < t \leq T$.*

In particular, for an arbitrary $T > 0$, there exists a constant $\epsilon_0(T)$ such that if $|P_0|, \|\langle x \rangle^4 \beta_0\|_2 \leq \epsilon_0(T)$, then P_t is bounded by

$$|P_t| \leq T^{-2}, \quad \text{for any } t \in [0, T]. \quad (3.10)$$

Proof. The local well-posedness of Eq. (3.8) is proven by standard techniques: One converts (3.8) into an integral equation for P_t that can be solved by iteration, as long as time t is small enough. The second part of Theorem 3.1 follows from the (proof of the) first part and the observation that if $P_0 = 0$ and $\beta_0 = 0$ then $P_t = 0$ is a global solution to (3.8). \square

4. Global Solution of (3.8)

In this section we establish **global well-posedness** of (3.8). We transform (3.8) to a more convenient form (see Eq. (4.14), below). For the new equation, we prove global existence of solutions by applying a fixed-point theorem on a suitably chosen Banach space of momentum trajectories.

To arrive at the new form of (3.8), we introduce a “propagator”, K , solving the following Wiener-Hopf equation:

$$\dot{K} = -Z \operatorname{Re} \langle W, \int_0^t e^{i \frac{\Delta}{2}(t-s)} K(s) ds W \rangle \quad (4.1)$$

with

$$K(0) = 1.$$

The constant Z is given by $Z := \frac{2}{3}\nu$, where $\nu > 0$ is the constant appearing in Eqs. (1.15) and (3.2). Some properties of the function $K(t)$ are described in the following proposition.

Proposition 4.1. *The function $K : \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuous and satisfies*

$$ZK(t) = \frac{1}{4}\pi^{-\frac{5}{2}}t^{-\frac{1}{2}} + C_K t^{-1} + O(t^{-\frac{3}{2}}), \quad (4.2)$$

as $t \rightarrow \infty$, for some constant $C_K \in \mathbb{R}$.

A detailed proof of this proposition forms the contents of Appendix A. On a formal level, the idea of the proof is straightforward. By taking Fourier transforms of both sides of (4.1) in the time variable t one obtains that

$$\int_0^\infty e^{ikt} K(t) dt = -\frac{1}{ik + ZG(k + i0)}, \quad (4.3)$$

where the function $G : \mathbb{R} \rightarrow \mathbb{C}$ is given by

$$G(k + i0) := -i \lim_{\epsilon \rightarrow 0^+} (\langle (-\Delta + 2k + i\epsilon)^{-1} W, W \rangle + i \langle (-\Delta - 2k - i\epsilon)^{-1} W, W \rangle). \quad (4.4)$$

Thus,

$$\begin{aligned} K(t) &= -(2\pi)^{-1} \int_{-\infty}^{\infty} \frac{1}{ik + ZG(k + i0)} e^{-ikt} dk \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Re} \frac{1}{ik + ZG(k + i0)} \cos kt \, dk. \end{aligned}$$

The function $G(k + i0)$ is smooth on $\mathbb{R} \setminus \{0\}$; in a neighborhood of $k = 0$ we have that $G(k + i0)$ is an analytic function of $k^{\frac{1}{2}}$, and there exists a constant $C \neq 0$ such that

$$G(k + i0) = Ck^{\frac{1}{2}} + O(|k|). \quad (4.5)$$

Formula (4.2) follows by finding the exact value for C .

Next, Lemma A.4 enables us to rewrite Eq. (3.8) in the following form:

$$P_t = K(t)P_0 + Z \int_0^t K(t-s) \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_0^s P_{s_1} ds_1 \, ds + \int_0^t K(t-s) F_P(s) \, ds. \quad (4.6)$$

In order to prove our main result on the decay of P_t in t , see inequality (2.2) of Theorem 2.1, we require more precise information on P_t than the one provided by (4.6). The problem is that all three terms on the right-hand side of (4.6) are $O(t^{-\frac{1}{2}})$, as $t \rightarrow \infty$. This decay is inadequate to prove inequality (2.2). We thus have to exhibit cancelations between the terms on the right-hand side of (4.6) that kill the leading terms. Our strategy to accomplish this is to resort to a second equation for P_t equivalent to (4.6) and then take a suitable linear combination of the two equations. To find the second equation we simply integrate both sides of (3.8) over time from 0 to t and arrive at

$$\begin{aligned} P_t &= P_0 + Z \int_0^t \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_0^s P_{s_1} ds_1 \, ds \\ &\quad - 2Z \operatorname{Re} \langle W, (i\Delta)^{-1} \int_0^t e^{i\frac{\Delta}{2}(t-s)} P_s \, ds \, W \rangle + \int_0^t F_P(s) \, ds, \end{aligned} \quad (4.7)$$

where the third term on the right-hand side of (4.7) derives from

$$-Z \int_0^t \operatorname{Re} \langle W, \int_0^s e^{i\frac{\Delta}{2}(s-s_1)} P_{s_1} ds_1 \, ds \, W \rangle$$

by integrating by parts in s and using that $\operatorname{Re} \langle W, (i\Delta)^{-1} W \rangle = 0$.

Multiplying both sides of (4.7) by $K(t)$ and subtracting the resulting equation from (4.6), we obtain that

$$\begin{aligned} [1 - K(t)]P_t &= Z \int_0^t [K(t-s) - K(t)] \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_0^s P_{s_1} ds_1 \, ds \\ &\quad + 2ZK(t) \operatorname{Re} \langle W, (i\Delta)^{-1} \int_0^t e^{i\frac{\Delta}{2}(t-s)} P_s \, ds \, W \rangle \\ &\quad + \int_0^t [K(t-s) - K(t)] F_P(s) \, ds. \end{aligned} \quad (4.8)$$

The first term on the right-hand side of (4.8) is rewritten as follows: for any function $q : [0, \infty) \rightarrow \mathbb{R}$,

$$\begin{aligned}
 & \int_0^t [K(t-s) - K(t)] \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle \int_0^s q_{s_1} ds_1 ds \\
 &= - \int_0^t [K(t-s) - K(t)] \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle \int_s^t q_{s_1} ds_1 ds \\
 & \quad + \int_0^t K(t-s) \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle ds \int_0^t q_{s_1} ds_1 \\
 & \quad - K(t) \int_0^t \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle ds \int_0^t q_{s_1} ds_1 \\
 &= - \int_0^t [K(t-s) - K(t)] \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle \int_s^t q_{s_1} ds_1 ds \\
 & \quad + \int_0^t K(t-s) \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle ds \int_0^t q_{s_1} ds_1 \\
 & \quad - 2K(t) \operatorname{Re} \langle W, (i\Delta)^{-1} e^{i \frac{\Delta}{2} t} W \rangle \int_0^t q_{s_1} ds_1.
 \end{aligned}$$

Plugging this expression into the right-hand side of (4.8), we find that

$$\begin{aligned}
 [1 - K(t)]P_t &= -Z \int_0^t [K(t-s) - K(t)] \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle \int_s^t P_{s_1} ds_1 ds \\
 & \quad + Z \int_0^t K(t-s) \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle ds \int_0^t P_{s_1} ds_1 \\
 & \quad + 2ZK(t) \operatorname{Re} \langle W, (i\Delta)^{-1} \int_0^t [e^{i \frac{\Delta}{2} (t-s)} - e^{i \frac{\Delta}{2} t}] P_s ds W \rangle \\
 & \quad + \int_0^t [K(t-s) - K(t)] F_P(s) ds \\
 &=: A(P)(t) + \int_0^t [K(t-s) - K(t)] F_P(s) ds, \tag{4.9}
 \end{aligned}$$

thus

$$P_t = \frac{1}{1 - K(t)} A(P)(t) + \frac{1}{1 - K(t)} \int_0^t [K(t-s) - K(t)] F_P(s) ds, \tag{4.10}$$

where A is the linear operator on the space of momentum trajectories $\{P_s\}_{0 \leq s < \infty}$ defined by (4.9).

Equation (4.10) has the desired form. We will show that, if the trajectories $\{P_s\}_{0 \leq s < \infty}$ belong to an appropriate Banach space, then a fixed-point theorem can be applied that implies global existence of solutions of (4.10). The main heuristic ideas underlying our approach are discussed in Appendix E.

Next, we introduce a family of Banach spaces of momentum trajectories: For an arbitrary $\delta \in I$, where I is the interval defined in (2.1), and any $T > 0$, we define the space

$$B_{\delta, T} := \{h : [T, \infty) \rightarrow \mathbb{R} \mid |t|^{\frac{1}{2} + \delta} h \in L^\infty[T, \infty)\} \tag{4.11}$$

equipped with the norm

$$\|h\|_{\delta,T} := \sup_{t \in [T, \infty)} t^{\frac{1}{2}+\delta} |h|(t). \quad (4.12)$$

The function $h(t)$ has the interpretation of being a component of P_t . The definition of $B_{\delta,T}$ can be extended to vector-valued functions $P_t : [T, \infty) \rightarrow \mathbb{R}^3$ in the obvious way.

Below, it will be proven that the operator A in (4.9) and (4.10) maps the space $B_{\delta,T}$ into itself, for T large enough; (see (4.2)). It appears to be difficult to show that $P_t \in B_{\delta,T}$, for some $\delta > 0$, by starting directly from (4.6). However, in the analysis of (4.10), a new difficulty appears: Since $K(0) = 1$, the operator $\frac{1}{1-K(t)}A(\cdot)(t)$ is unbounded on $L^\infty[0, \infty)$. The new difficulty is circumvented by “waiting long enough until $K(t)$ becomes small”. We therefore divide the time axis $[0, \infty)$ into two subintervals, $[0, T]$ and $[T, \infty)$, where T is chosen such that $|K(t)| \ll 1$ when $t \geq T$. For $t \in [0, T]$, a unique strong solution, P_t , to the equation of motion (3.8) corresponding to a given initial condition $P_{t=0} = P_0$ exists, provided $|P_0|$ and β_0 are small enough, depending on T , as shown in Theorem 3.1. For $t \in [T, \infty)$, we show that a solution P_t exists and belongs to the space $B_{\delta,T}$ by proving that the operator $\frac{1}{1-K(t)}A : B_{\delta,T} \rightarrow B_{\delta,T}$ is a contraction on a sufficiently small ball in $B_{\delta,T}$ centered at the origin; evidently this forces us to require that $|P_T|$ is small enough, which, by Theorem 3.1, is guaranteed if $|P_0|$ and $\|\langle x \rangle^4 \beta_0\|_2$ are chosen to be sufficiently small.

In what follows, we convert this discussion into rigorous mathematics. We define

$$\chi_T(t) := \begin{cases} 1 & \text{if } 0 \leq t < T \\ 0 & \text{if } t \geq T, \end{cases} \quad (4.13)$$

and rewrite Eq. (4.10) as

$$P_t = \Upsilon(P)(t) + G(t), \quad (4.14)$$

where $G(t)$ is the contribution to the right-hand side of (4.10) only depending on $\{P_t\}_{0 \leq t < T}$, i.e., independent of $(1 - \chi_T)P$,

$$G(t) := \frac{1}{1-K(t)} \{A(\chi_T P_t) + \int_0^t [K(t-s) - K(t)] F_{\chi_T P}(s) ds\}, \quad (4.15)$$

and $\Upsilon(P)$ contains terms of first order in $(1 - \chi_T)P$ and higher order in P and is given by

$$\begin{aligned} \Upsilon(P)(t) &:= \frac{1}{1-K(t)} \{A((1 - \chi_T)P) \\ &\quad + \int_0^t [K(t-s) - K(t)] [F_P(s) - F_{\chi_T P}(s)] ds\}. \end{aligned} \quad (4.16)$$

In Theorem 3.1, we have shown that if the initial conditions P_0 and β_0 are sufficiently small ($|P_0|, \|\langle x \rangle^4 \beta_0\|_2 \leq \epsilon_0(T)$) then there exists a unique solution P_t , $t \in [0, T]$, with $\max_{t \in [0, T]} |P_t| \leq T^{-2}$. In order to continue this solution to the interval $[T, \infty)$ and to show that $\{P_t\}_{T \leq t < \infty}$ belongs to the Banach space $B_{\delta,T}$ we propose to use a fixed-point theorem, which can be applied, provided two conditions are fulfilled:

- (1) The nonlinear map $\Upsilon(\cdot) + G$ maps a small ball, \mathcal{B} , in the Banach space $B_{\delta,T}$ centered at 0 into itself, in particular, $\|G\|_{\delta,T}$ is small enough, and

(2) $\Upsilon(\cdot)$ is a contraction on \mathcal{B} in the norm $\|\cdot\|_{\delta,T}$ of $B_{\delta,T}$ introduced in (4.12).

We begin by verifying that $\Upsilon(\cdot)$ is a contraction on \mathcal{B} if \mathcal{B} is chosen small enough. We define a function $\Omega : [0, 1] \rightarrow \mathbb{R}^+$ by

$$\Omega(\delta) := \int_0^1 \frac{1}{1 + (1-r)^{\frac{1}{2}}} (1-r)^{-\frac{1}{2}} \left[\frac{1}{1-2\delta} (r^{-\frac{1}{2}} - r^{-\delta}) + r^{\frac{1}{2}-\delta} \right] dr, \quad (4.17)$$

(see Eq. (2.1)). A key result is the following theorem.

Theorem 4.2. *There exists a constant $T_0 < \infty$ such that if $T \geq T_0$ and if δ is chosen such that $\pi^{-1}\Omega(\delta) < 1$ then the map $\Upsilon(\cdot)$ introduced in (4.16) maps $B_{\delta,T}$ into itself and is a contraction on a sufficiently small ball $\mathcal{B} \subset B_{\delta,T}$ centered at 0. The two terms on the right-hand side of (4.16) defining $\Upsilon(\cdot)$ satisfy the following estimates:*

(1) *The linear operator $A((1 - \chi_T)\cdot)$ satisfies*

$$\left| \frac{1}{1-K(t)} A((1 - \chi_T)h)(t) \right| \leq t^{-\frac{1}{2}-\delta} \left[\frac{1}{\pi} \Omega(\delta) + \epsilon(T) \right] \|h\|_{\delta,T}, \quad (4.18)$$

where $\epsilon(T)$ is a small constant satisfying $\lim_{T \rightarrow \infty} \epsilon(T) = 0$.

(2) *Let $\epsilon_0(T)$ be the constant introduced in Theorem 3.1. Suppose that $Q_1, Q_2 : [0, \infty) \rightarrow \mathbb{R}^3$ are any two vector-valued functions satisfying the following two conditions:*

$$Q_1(t) = Q_2(t) = P_t, \quad t \in [0, T],$$

where P_t is the solution of Eq. (3.8)/(4.10) constructed in Theorem 3.1, for $0 \leq t \leq T$ and $|P_0|, \|\langle x \rangle^4 \beta_0\|_2 \leq \epsilon_0(T)$; and

$$\|Q_1\|_{\delta,T}, \|Q_2\|_{\delta,T} \ll 1.$$

Then

$$\begin{aligned} & \left| \frac{1}{1-K(t)} \int_0^t [K(t-s) - K(t)] [F_{Q_1} - F_{Q_2}](s) ds \right| \\ & \lesssim t^{-\frac{1}{2}-\delta} \|Q_1 - Q_2\|_{\delta,T} [\|Q_1\|_{\delta,T} + \|Q_2\|_{\delta,T}]. \end{aligned} \quad (4.19)$$

Inequality (4.18) will be reformulated as Proposition 4.4, below, and proven in Appendix B. Inequality (4.19) is proven in Appendix C.

Next, we present an estimate on the term $G(t)$ on the right-hand side of (4.14) defined in (4.15). This term only depends on the solution, P_t , of (3.8)/(4.10) for $t \in [0, T]$, which has been constructed in Theorem 3.1.

Theorem 4.3. *Suppose that $|P_0|, \|\langle x \rangle^4 \beta_0\|_2 \leq \epsilon_0(T)$ and that the parameter δ is chosen as in Theorem 4.2. Then G belongs to the Banach space $B_{\delta,T}$ and $\|G\|_{\delta,T}$ can be made arbitrarily small by choosing T large enough; (see Theorem 3.1).*

More specifically, we have that, for any $t \geq T$,

$$\left| \frac{1}{1-K(t)} \int_0^t [K(t-s) - K(t)] F_{\chi_T P}(s) ds \right| \leq \epsilon(T) t^{-\frac{1}{2}-\delta}, \quad (4.20)$$

and

$$\left| \frac{1}{1-K(t)} A(\chi_T P_t) \right| \leq \epsilon(T) t^{-\frac{1}{2}-\delta}, \quad (4.21)$$

with $\epsilon(T) \rightarrow 0$ as $T \rightarrow \infty$.

Remark 2. The constant $\epsilon_0(T)$ is chosen as in Theorem 3.1.

The proof of this theorem is contained in Appendix D.

In the remainder of this section, we discuss the strategy used to prove Theorem 4.2. (The proof of Theorem 4.3 is easier than the one of Theorem 4.2 and is therefore not discussed here.) We recall that the map $\Upsilon(\cdot)$ is the sum of two maps appearing on the right-hand side of (4.16). The map $A((1 - \chi_T)\cdot)(t)$ is linear, while $\frac{1}{1-K(t)} \int_0^t [K(t-s) - K(t)][F_P(s) - F_{\chi_T P}(s)] ds$ contains higher-order terms, besides depending on the initial conditions, β_0 , of the Bose gas. Since we are attempting to construct **small** solutions of (3.8)/(4.10), and because β_0 can be chosen as small as needed, it is fairly easy to control the second map. We therefore focus our attention on the ideas needed to estimate the first (linear) map. For this purpose, we write the operator $A((1 - \chi_T)\cdot)$ as a sum of three terms:

$$A((1 - \chi_T)h)(t) = \sum_{k=1}^3 \Gamma_k(t), \quad \Gamma_k(t) \equiv \Gamma_k((1 - \chi_T)h)(t), \quad (4.22)$$

where

$$\begin{aligned} \Gamma_1(t) &:= -Z \int_0^t ds [K(t-s) - K(t)] \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^t ds_1 h_{s_1} [1 - \chi_T(s_1)], \\ \Gamma_2(t) &:= Z \int_0^t ds K(t-s) \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_0^t ds_1 h_{s_1} [1 - \chi_T(s_1)], \end{aligned}$$

and

$$\Gamma_3(t) := 2Z K(t) \operatorname{Re} \langle W, (i\Delta)^{-1} \int_0^t ds [e^{i\frac{\Delta}{2}(t-s)} - e^{i\frac{\Delta}{2}t}] h_s [1 - \chi_T(s)] W \rangle, \quad (4.23)$$

see Eqs. (4.9) and (4.13).

Before estimating $\Gamma_k(t)$, $k = 1, 2, 3$, we introduce two functions:

$$\Omega_1(\delta) := \frac{1}{(1-2\delta)\pi} \int_0^1 \frac{1}{1+(1-r)^{\frac{1}{2}}} (1-r)^{-\frac{1}{2}} [r^{-\frac{1}{2}} - r^{-\delta}] dr, \quad (4.24)$$

and

$$\Omega_2(\delta) := \frac{1}{\pi} \int_0^1 \frac{1}{1+(1-r)^{\frac{1}{2}}} (1-r)^{-\frac{1}{2}} r^{\frac{1}{2}-\delta} dr,$$

see also (4.17).

Control of the terms $\Gamma_k(t)$, $k = 1, 2, 3$, is provided in the following proposition.

Proposition 4.4. *There exists a function $\tilde{\epsilon}(T)$, $0 < T < \infty$, with $\lim_{T \rightarrow \infty} \tilde{\epsilon}(T) = 0$, such that, for an arbitrary function $h \in B_{\delta,T}$,*

$$|\Gamma_1(t)| \leq t^{-\frac{1}{2}-\delta} [\Omega_1(\delta) + \tilde{\epsilon}(T)] \|h\|_{\delta,T}, \quad (4.25)$$

$$|\Gamma_3(t)| \leq t^{-\frac{1}{2}-\delta} [\Omega_2(\delta) + \tilde{\epsilon}(T)] \|h\|_{\delta,T}, \quad (4.26)$$

and

$$|\Gamma_2(t)| \leq \tilde{\epsilon}(T) t^{-\frac{1}{2}-\delta} \|h\|_{\delta,T}, \quad (4.27)$$

where $B_{\delta,T}$ and $\|\cdot\|_{\delta,T}$ are as in (4.11) and (4.12) respectively.

This proposition will be proven in Appendix B. Obviously it implies inequality (4.18) in Theorem 4.2, (1), with $\epsilon(T) = 3\tilde{\epsilon}(T)$. The ideas underlying the proof of Proposition 4.4 are as follows. The arguments needed to estimate $\Gamma_1(t)$ and $\Gamma_3(t)$ are very similar, so we only consider $\Gamma_1(t)$ and $\Gamma_2(t)$. The trickiest estimate is (4.27). The crucial step is to control the factor

$$I(t) := \int_0^t K(t-s) Z \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle ds$$

in $\Gamma_2(t)$. We propose to show that

$$I(t) = O((1+t)^{-\frac{3}{2}}), \quad \text{as } t \rightarrow \infty, \quad (4.28)$$

which implies (4.27) by straightforward arguments.

Estimate (4.28) does not follow by just using that $K(t) = O(t^{-\frac{1}{2}})$ and $\langle W, e^{i\frac{\Delta}{2}t} W \rangle = O((1+t)^{-\frac{3}{2}})$. These estimates, by themselves, only imply that $|I(t)| \leq \text{const } t^{-\frac{1}{2}}$. In order to conclude the improved estimate claimed in (4.28), we Fourier-transform the convolution of K with $\operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle \chi_0$, with $\chi_0(k) = 1$, for $k \geq 0$, and $= 0$, otherwise, which yields

$$I(t) = \frac{Z}{2\pi} \int_{-\infty}^{\infty} \widehat{K}(k) \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle \chi_0(k) e^{-ikt} dk.$$

From (4.1), (4.3) and (4.4) we derive by inspection that $\widehat{\operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle \chi_0}(k) = -G(k+i0)$, hence

$$I = -\frac{Z}{2\pi} \int_{-\infty}^{\infty} \frac{G(k+i0)}{ik + ZG(k+i0)} e^{-ikt} dk, \quad (4.29)$$

with $G(k+i0)$ as in (4.4). The function $\frac{G(k+i0)}{ik+ZG(k+i0)}$ is smooth in k on $\mathbb{R} \setminus \{0\}$. It is therefore its behavior near $k = 0$ that determines the decay of $I(t)$ in t . We recall that $G(k+i0) = ck^{\frac{1}{2}} + O(k)$, for $|k|$ small, where c is some non-zero constant; see (4.5). Thus

$$-Z \frac{G(k+i0)}{ik + ZG(k+i0)} = -1 + \frac{1}{ic} k^{\frac{1}{2}} + O(k), \quad \text{for } |k| \text{ small.} \quad (4.30)$$

Furthermore $\frac{G(k+i0)}{ik+ZG(k+i0)}$ decays rapidly in k at infinity. Thus, in Eq. (4.29), we can integrate by parts in the variable k , and this yields the desired decay estimate on $I(t)$; (note that the constant term on the right-hand side of (4.30) yields a subleading contribution).

Next, we turn to estimating Γ_1 ; see (4.25). For this purpose, we peel off the main contributions to the functions $K(t)$ and $\operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle$: By explicit calculation, see (4.2) and (A.26), one finds that, as $t \rightarrow \infty$, $K(t) = \frac{1}{4Z} \pi^{-\frac{5}{2}} t^{-\frac{1}{2}} + O(t^{-1})$; see (4.2), and $\operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle = C_W t^{-\frac{3}{2}} + O((1+t)^{-\frac{5}{2}})$. We define an approximation, $\tilde{\Gamma}_1(t)$, of $\Gamma_1(t)$ by

$$\tilde{\Gamma}_1 := -\frac{1}{4} \pi^{-\frac{5}{2}} C_W \int_0^t ds [(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}] s^{-\frac{3}{2}} \int_s^t ds_1 h_{s_1} [1 - \chi_T(s_1)].$$

Recalling the definition of the Banach space $\mathcal{B}_{\delta,T}$ and changing variables, $s = t\sigma$ and $s_1 = t\sigma_1$, we find that

$$\begin{aligned} |\tilde{\Gamma}_1| &\leq t^{-\frac{1}{2}-\delta} \frac{1}{4} C_W \int_0^1 [(1-\sigma)^{-\frac{1}{2}} - 1] \sigma^{-\frac{3}{2}} \int_\sigma^1 \sigma_1^{-\frac{1}{2}-\delta} d\sigma_1 d\sigma \|h\|_{\delta,T} \\ &\leq t^{-\frac{1}{2}-\delta} \Omega_1(\delta) \|h\|_{\delta,T}. \end{aligned}$$

To complete our estimate on Γ_1 we are left with estimating $(\Gamma_1 - \tilde{\Gamma}_1)(t)$, which is a straightforward task. In fact $\Gamma_1(t) - \tilde{\Gamma}_1(t)$ decays in t faster than $\tilde{\Gamma}_1(t)$.

Further details of our estimates needed to prove Proposition 4.4 can be found in Subsect. B.2 of Appendix B.

5. Proof of the Main Result, Theorem 2.1

We use the subdivision of the time axis

$$[0, \infty) = [0, T) \cup [T, \infty)$$

into two parts, for an appropriately chosen T . Existence and uniqueness of a solution P_t , of Eqs. (3.8)/(4.10), for $t \in [0, T)$, assuming that the initial conditions P_0 and β_0 are small enough (depending on T), has been proven in Theorem 3.1.

To continue such a solution to the interval $[T, \infty)$, we apply a standard fixed-point theorem to Eq. (4.14). Thanks to Theorems 4.2 and 4.3, the hypotheses of the fixed-point theorem are valid, provided T is chosen appropriately, and $|P_0|$, $\|\langle x \rangle^4 \beta_0\|_2$ are small enough. We thus conclude that a global solution P_t , $t \in [0, \infty)$, to (3.8)/(4.10) exists, with $(1 - \chi_T)P_t \in \mathcal{B}_{\delta,T}$, for any δ in the interval I defined in (2.1), provided $|P_0|$ and $\|\langle x \rangle^4 \beta_0\|_2$ are chosen small enough. This proves estimate (2.2) of Theorem 2.1.

In order to prove Eq. (2.3) in Theorem 2.1, we show that the field δ_t introduced in (3.1) decays to 0, as $t \rightarrow \infty$, in the sense that

$$\|\delta_t\|_\infty \rightarrow 0, \quad \text{as } t \rightarrow \infty. \quad (5.1)$$

To establish (5.1) we apply the norm $\|\cdot\|_\infty$ to both sides of (3.4), which yields

$$\begin{aligned} \|\delta_t\|_\infty &\leq \text{const} \cdot \{ \|e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} W^{X_0}\|_\infty + \|e^{i\frac{\Delta}{2}t} \beta_0\|_\infty \\ &\quad + \int_0^t \|e^{i\frac{\Delta}{2}s} (-\Delta)^{-1} P_s \nabla_x W^{X_s}\|_\infty ds \}. \end{aligned}$$

Using the estimates

$$\|e^{i\frac{\Delta}{2}t} (-\Delta)^{-1}\|_{L^1 \rightarrow L^\infty} \leq \text{const } t^{-\frac{1}{2}}$$

and

$$\|e^{i\frac{\Delta}{2}t}\|_{L^1 \rightarrow L^\infty} \leq \text{const } t^{-\frac{3}{2}}$$

and our estimate on P_t , see (2.2), we find that

$$\|\delta_t\|_\infty \leq \text{const } t^{-\frac{1}{2}} + t^{-\frac{3}{2}} \|\beta_0\|_{L^1}$$

which, under our assumption on β_0 , yields (5.1).

This completes the proof of Theorem 2.1. Some hard technicalities now follow in several appendices. \square

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A. Proof of Proposition 4.1

We start with deriving an explicit formula for K , see (4.1).

We define a function $G : \mathbb{R} \rightarrow \mathbb{C}$ by

$$G(k+i0) := -i \langle (-\Delta + 2k + i0)^{-1} W, W \rangle + i \langle (-\Delta - 2k - i0)^{-1} W, W \rangle. \quad (\text{A.1})$$

Next, we relate G to the function K .

Proposition A.1. *The function K in (4.1) is continuous and takes the form*

$$K(t) = -(2\pi)^{-1} \int_{-\infty}^{\infty} \frac{1}{ik + ZG(k+i0)} e^{-ikt} dk, \quad (\text{A.2})$$

in particular

$$K(t) = 0 \text{ for } t < 0. \quad (\text{A.3})$$

The function K can be transformed to a convenient form

$$K(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Re} \frac{1}{ik + ZG(k+i0)} \cos kt \, dk. \quad (\text{A.4})$$

This proposition is proven in Subsects. A.2 and A.3. The basic ideas in the proof are not difficult. On a formal level, Eq. (A.2) is obtained by Fourier transformation, as mentioned after Proposition 4.1. In Subsect. A.3 we will make it rigorous. Equation (A.3) follows from the facts that $\frac{1}{iz + ZG(z)}$ is analytic in the set $\operatorname{Im} z > 0$ and its absolute value is sufficiently small when $|z|$ is large. Hence

$$K(t) = -(2\pi)^{-1} \int_{-\infty}^{\infty} \frac{1}{i(k+ia) + ZG(k+a)} e^{-i(k+ia)t} dk$$

for any $a > 0$, and moreover $K(t) \rightarrow 0$, as $a \rightarrow \infty$, if $t < 0$. We obtain the last identity (A.4) by manipulating the expression in (A.2).

To prove Proposition 4.1 it suffices to derive a decay estimate for $K(t)$ from (A.4) using the oscillatory nature of $\cos kt$. Since the function $\operatorname{Re} \frac{1}{ik + ZG(k+i0)} : \mathbb{R} \rightarrow \mathbb{R}$ is smooth on the open set $(-\infty, \infty) \setminus \{0\}$, it is the lowest order term in the Taylor-expansion of the function in a neighborhood of $k = 0$ that determines the decay in t .

Lemma A.2. *The function $G(k+i0)$ defined in (A.1) satisfies the estimate*

$$G(k+i0) = \begin{cases} 2^{\frac{3}{2}}(i-1)\pi^2 k^{\frac{1}{2}} + Ck + O(|k|^{\frac{3}{2}}) & \text{if } k > 0 \\ 2^{\frac{3}{2}}(-i-1)\pi^2 |k|^{\frac{1}{2}} + Ck + O(|k|^{\frac{3}{2}}) & \text{if } k < 0, \end{cases} \quad (\text{A.5})$$

where C is some constant.

This lemma is proven in Subsect. A.1, by Taylor-expanding the function $G(k+i0)$ in $k^{\frac{1}{2}}$.

Now we are ready to prove Proposition 4.1:

Proof of Proposition 4.1. The fact K is a continuous function has been shown in Proposition A.1.

For the other parts, to simplify matters we decompose $K(t)$ into two parts, according to the integration regions

$$K(t) = K_+(t) + K_-(t), \quad (\text{A.6})$$

with

$$K_+ := -\frac{1}{\pi} \int_0^\infty \operatorname{Re} \frac{1}{ik + ZG(k+i0)} \cos kt \, dk$$

and

$$K_- := -\frac{1}{\pi} \int_{-\infty}^0 \operatorname{Re} \frac{1}{ik + ZG(k+i0)} \cos kt \, dk.$$

We first estimate K_+ . Since the leading order is determined by Taylor expansion of the integrand $\operatorname{Re} \frac{1}{ik + ZG(k+i0)} \cos kt$ around $k = 0$; it is natural to begin with studying this function. We define a new function $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ by

$$|k|^{-\frac{1}{2}} g(|k|^{\frac{1}{2}}) := \operatorname{Re} - \frac{1}{\pi} \frac{1}{ik + ZG(k+i0)}.$$

By direct computation, using the result in (A.5), we find that, in a neighborhood of $k = 0$,

$$|k|^{-\frac{1}{2}} g(|k|^{\frac{1}{2}}) = -\frac{1}{\pi Z} \frac{\operatorname{Re} G}{(\frac{k}{Z} + \operatorname{Im} G)^2 + (\operatorname{Re} G)^2} = \frac{1}{2^{\frac{5}{2}} \pi^3 Z} |k|^{-\frac{1}{2}} [1 + O(k^{\frac{1}{2}})],$$

where, in the last step, the result in (A.5) was used. The other important observations are that the function $g : \mathbb{R}^+ \rightarrow \mathbb{C}$ is smooth on $[0, \infty)$ and satisfies the estimate

$$|g(\rho)| \leq C(1 + \rho)^{-3}.$$

Expanding $g(k)$ around $k = 0$ we obtain

$$\begin{aligned} K_+(t) &= \int_0^\infty |k|^{-\frac{1}{2}} g(|k|^{\frac{1}{2}}) \cos kt \, dk \\ &= 2 \int_0^\infty g(\rho) \cos(\rho^2 t) \, d\rho \\ &= 2g(0) \int_0^\infty \cos(\rho^2 t) \, d\rho + D, \end{aligned}$$

where D is given by

$$D := 2 \int_0^\infty [g(\rho) - g(0)] \cos(\rho^2 t) \, d\rho.$$

The first term on the right-hand side is the dominant one,

$$2g(0) \int_0^\infty \cos(\rho^2 t) \, d\rho = 2g(0) t^{-\frac{1}{2}} \int_0^\infty \cos x^2 \, dx = \frac{1}{8} \pi^{-\frac{5}{2}} Z^{-1} t^{-\frac{1}{2}}, \quad (\text{A.7})$$

using the Fresnel integral (A.30).

The second term, D , is of the form

$$D = \int_0^\infty H(\rho) \cos(\rho^2 t) \rho \, d\rho,$$

where the function $H = \frac{2(g(\rho) - g(0))}{\rho}$ is smooth and is bounded uniformly by $C(1 + \rho)^{-1}$. By standard techniques we find that

$$D = O(t^{-\frac{3}{2}}). \quad (\text{A.8})$$

This together with (A.7) implies that

$$K_+ = \frac{1}{8} \pi^{-\frac{5}{2}} Z^{-1} t^{-\frac{1}{2}} + O(t^{-\frac{3}{2}}). \quad (\text{A.9})$$

For K_- , we obtain, using almost identical arguments,

$$K_- = \frac{1}{8} \pi^{-\frac{5}{2}} Z^{-1} t^{-\frac{1}{2}} + O(t^{-\frac{3}{2}}).$$

These two results, together with (A.6), obviously imply Proposition 4.1. \square

In the next three subsections we prove Lemma A.2 and Proposition A.1.

In what follows we often use Fourier transform. Its definition and properties are standard. Since constants are important in the present paper, we quote some of them explicitly. For a function $f : \mathbb{R}^d \rightarrow \mathbb{C}$, its Fourier transformation \hat{f} is defined as

$$\hat{f}(k) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ik \cdot x} f(x) \, dx,$$

and the inverse transform as

$$\check{f}(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ik \cdot x} f(k) \, dk.$$

Moreover, for arbitrary functions $f, g : \mathbb{R}^d \rightarrow \mathbb{C}$,

$$\widehat{fg} = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \hat{f}(x - y) \hat{g}(y) \, dy. \quad (\text{A.10})$$

A.1. Proof of Lemma A.2.

Proof. By Fourier-transformation and introducing polar coordinates we find that $G(k + i0)$ takes the convenient form

$$\begin{aligned} G(k + i0) &= -i \langle (\rho^2 + 2k + i0)^{-1} \hat{W}, \hat{W} \rangle + i \langle (\rho^2 - 2k - i0)^{-1} \hat{W}, \hat{W} \rangle \\ &= i8\pi k \left[\int_0^\infty (\rho^2 + 2k + i0)^{-1} |\hat{W}(\rho)|^2 d\rho \right. \\ &\quad \left. + \int_0^\infty (\rho^2 - 2k - i0)^{-1} |\hat{W}(\rho)|^2 d\rho \right]. \end{aligned} \quad (\text{A.11})$$

We expand $|\hat{W}(\rho)|^2$ in a neighborhood of $\rho = 0$. The fact that the function $W : \mathbb{R}^3 \rightarrow \mathbb{R}$ is smooth, spherically symmetric and decays rapidly at ∞ implies that the

function \hat{W} is smooth in the variable ρ^2 , or, equivalently, that there exists a smooth and rapidly decaying function $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$|\hat{W}(\rho)|^2 = 1 + \rho^2 F(\rho^2). \quad (\text{A.12})$$

Here the condition that $|\hat{W}(0)| = 1$, in (1.17), is used. Plugging (A.12) into (A.11) we find that

$$G(k + i0) = G_0(k) + G_1(k) + G_2(k)$$

with

$$\begin{aligned} G_0(k + i0) &:= i8\pi k \left[\int_0^\infty (\rho^2 + 2k + i0)^{-1} d\rho + \int_0^\infty (\rho^2 - 2k - i0)^{-1} d\rho \right], \\ G_1(k + i0) &:= 8i\pi k \int_0^\infty F(\rho^2) d\rho, \end{aligned}$$

and

$$\begin{aligned} G_2(k + i0) &:= -16i\pi k^2 \left[\int_0^\infty (\rho^2 + 2k + i0)^{-1} F(\rho^2) d\rho \right. \\ &\quad \left. - \int_0^\infty (\rho^2 - 2k - i0)^{-1} F(\rho^2) d\rho \right]. \end{aligned}$$

Next we estimate the three terms.

The estimate on G_1 is evident:

$$G_1(k + i0) = O(k). \quad (\text{A.13})$$

The function $G_0(k + i0)$ has an explicit expression: For any $k \in \mathbb{C} \setminus \mathbb{R}^-$, we observe that

$$\int_0^\infty (\rho^2 + 2k)^{-1} d\rho = \frac{\pi}{2\sqrt{2}} k^{-\frac{1}{2}}, \quad (\text{A.14})$$

where $k^{-\frac{1}{2}} = |k|^{-\frac{1}{2}}$, for $k \in \mathbb{R}^+$. The proof consists in observing that if $k > 0$ then

$$\int_0^\infty (\rho^2 + 2k)^{-1} d\rho = \frac{1}{\sqrt{2}} k^{-\frac{1}{2}} \int_0^\infty (\rho^2 + 1)^{-1} d\rho = \frac{\pi}{2\sqrt{2}} k^{-\frac{1}{2}}.$$

Consequently

$$G_0(k + i0) = \begin{cases} 2^{\frac{3}{2}}(i - 1)\pi^2 k^{\frac{1}{2}} & \text{if } k > 0 \\ 2^{\frac{3}{2}}(-i - 1)\pi^2 |k|^{\frac{1}{2}} & \text{if } k < 0. \end{cases} \quad (\text{A.15})$$

To estimate G_2 it is sufficient to show that if a function ϕ decays sufficiently fast at ∞ then, for any small k ,

$$\int_{-\infty}^\infty (\rho^2 \pm 2k \pm i0)^{-1} \phi(\rho) d\rho = O(|k|^{-\frac{1}{2}}). \quad (\text{A.16})$$

Indeed, we use Fourier transformation to relate $(\rho^2 \pm 2k \pm i0)^{-1}$ to $(-\partial_x^2 \pm 2k \pm i0)^{-1}$ and find that

$$(-\partial_x^2 \pm 2k \pm i0)^{-1} \hat{\phi}|_{x=0} = C \int_{-\infty}^{\infty} (\rho^2 \pm 2k \pm i0)^{-1} \phi(\rho) d\rho$$

for some constant $C \neq 0$. The operator $(-\partial_x^2 + 2k)$, $k \in \mathbb{C} \setminus \mathbb{R}^-$, has an integral kernel $C_1 k^{-\frac{1}{2}} e^{-k^{\frac{1}{2}}|x-y|}$, where C_1 is a constant. This yields

$$|\int_{-\infty}^{\infty} (\rho^2 \pm 2k \pm i0)^{-1} \phi(\rho) d\rho| \lesssim |k|^{-\frac{1}{2}} \|\phi\|_{L^1},$$

which is the desired estimate (A.16).

Collecting the estimates above we complete our proof. \square

A.2. Proof of (A.3) of Proposition A.1. We start by extending the domain of the function G from $k \in \mathbb{R}$ to $\text{Im } k > 0$. By Fourier transformation we find that $G(k + i0)$ takes the form

$$G(k + i0) = -i \langle (\rho^2 + 2k + i0)^{-1} \hat{W}, \hat{W} \rangle + i \langle (\rho^2 - 2k - i0)^{-1} \hat{W}, \hat{W} \rangle. \quad (\text{A.17})$$

Its extension is the function $G : \{k | \text{Im } k > 0\} \rightarrow \mathbb{C}$ defined by

$$G(k) := 8i\pi k \left[\int_0^{\infty} (\rho^2 + 2k)^{-1} |\hat{W}(\rho)|^2 d\rho + \int_0^{\infty} (\rho^2 - 2k)^{-1} |\hat{W}(\rho)|^2 d\rho \right].$$

It is easy to see that this function is analytic in k , $\text{Im } k > 0$, and $G(k + i0)$, $k \in \mathbb{R}$, is its limit on the real line.

Equation (A.3) follows by contour integration. To guarantee its applicability, we have to verify several criteria.

We start with the following result.

Lemma A.3. *In the complex region $\text{Im } k > 0$, the function $\frac{1}{ik + ZG(k)} e^{ikt}$ is analytic in k for any (fixed) t , and*

$$\lim_{|k| \rightarrow \infty} \left| \frac{1}{ik + ZG(k)} \right| \rightarrow 0. \quad (\text{A.18})$$

Proof. Equation (A.18) is implied by the fact that $G(k) \rightarrow 0$, as $|k| \rightarrow \infty$.

It is easy to see that the function $ik + ZG(k)$ is analytic, in the region $\text{Im } k > 0$, because the operator $(-\Delta \pm 2k)^{-1}$ is well defined and analytic in k . To prove analyticity of $e^{ikt} \frac{1}{ik + ZG(k)}$ we only need to prove that the denominator does vanish anywhere, i.e.

$$|ik + ZG(k)| \neq 0 \quad \text{when } \text{Im } k > 0. \quad (\text{A.19})$$

For this purpose we rewrite the expression of $G(k)$ in (A.24) to obtain

$$ik + ZG(k) = ik \left[1 + 8Z\pi \int_0^{\infty} \rho^2 (\rho^4 - 4k^2) |\hat{W}(\rho)|^2 d\rho \right]. \quad (\text{A.20})$$

In what follows we consider two cases, $\text{Re } k = 0$ and $\text{Re } k \neq 0$ separately.

- (A) If $\operatorname{Re} k = 0$ then, by the fact that $\operatorname{Im} k > 0$, we find that $-k^2 > 0$, hence $\rho^4 - 4k^2 > 0$, and this implies (A.19).
- (B) If $\operatorname{Re} k \neq 0$, the key observation is that $\int_0^\infty \rho^2(\rho^4 - 4k^2)|\hat{W}(\rho)|^2 d\rho$ has a non-vanishing imaginary part. Indeed $-4k^2$ can be written in the form $-4k^2 = a + ib$, with $b \neq 0$. We rewrite $(\rho^4 - 4k^2)^{-1}$ as $(\rho^4 - 4k^2)^{-1} = [(\rho^4 + a)^2 + b^2]^{-1}(\rho^4 + a - ib)$. Hence

$$\operatorname{Im} \int_0^\infty \rho^2(\rho^4 - 4k^2)|\hat{W}(\rho)|^2 d\rho = -b \int_0^\infty [(\rho^4 + a)^2 + b^2]^{-1} |\hat{W}(\rho)|^2 d\rho \neq 0.$$

This together with (A.20) implies (A.19). \square

We continue to prove (A.3). The fact that $\frac{1}{ik + ZG(k)} e^{ikt}$ is analytic on the domain $\operatorname{Im} k > 0$ and the decay estimate in (A.18), proven in Lemma A.3, imply that, for any $a > 0$,

$$\begin{aligned} F(t) &:= \int_{-\infty}^{\infty} \frac{1}{ik + ZG(k + i0)} e^{-ikt} dk \\ &= \int_{-\infty}^{\infty} \frac{1}{i(k + ia) + ZG(k + ia)} e^{-i(k+ia)t} dk \\ &= e^{at} \int_{-\infty}^{\infty} \frac{1}{i(k + ia) + ZG(k + ia)} e^{-ikt} dk. \end{aligned}$$

If $t < 0$ it is easy to see that $F(t) = 0$ by letting $a \rightarrow +\infty$. This is (A.3).

We turn to (A.4). The definition of $G(k + i0)$ implies that $G(k + i0) = -G(-k - i0)$. Moreover, by changing variables $k \rightarrow -k$, we find that, for any $t > 0$,

$$F(t) = \int_{-\infty}^{\infty} \frac{1}{-ik - ZG(k - i0)} e^{ikt} dk$$

and

$$F(-t) = \int_{-\infty}^{\infty} \frac{1}{-ik - ZG(k - i0)} e^{-ikt} dk = 0.$$

These identities, together with (A.3) and the observation that $\overline{G(k + i0)} = -G(k - i0)$, imply

$$\begin{aligned} F(t) &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{1}{ik + ZG(k + i0)} + \frac{1}{-ik - ZG(k - i0)} \right] [e^{ikt} + e^{-ikt}] dk \\ &= 2 \int_{-\infty}^{\infty} \operatorname{Re} \frac{1}{ik + ZG(k + i0)} \cos kt dk. \end{aligned}$$

The desired result follows by using the definition of $F(t)$. \square

A.3. Proof of (A.2) of Proposition A.1. The idea underlying the proof is simple, at least on a formal level: By Fourier transforming both sides of (4.1) one obtains an explicit expression for $\int_0^\infty e^{ikt} q_t dt$. Then, by inverse Fourier transformation, one arrives at the desired result. The setback is that the Fourier transformation and its inverse are defined on L^2 , and the fact that the function $K \notin L^2$ makes things a little harder. In what follows, we begin with another result and show that it implies (4.1).

The result is

Lemma A.4. *Suppose the function $\tilde{q}_t : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies the estimate $|\tilde{q}_t| \leq C_1(1+t)^{-2}$ and is the solution to the equation*

$$\partial_t \tilde{q}_t = -Z \operatorname{Re} \langle W, \int_0^t e^{i \frac{\Delta}{2}(t-s)} \tilde{q}_s ds W \rangle + h(t), \quad (\text{A.21})$$

where the function $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ decays like $|h(t)| \leq c_2(1+t)^{-\frac{3}{2}}$. Then \tilde{q}_t also satisfies the equation

$$\tilde{q}_t = K(t) \tilde{q}_0 + \int_0^t K(t-s) h(s) ds. \quad (\text{A.22})$$

This lemma will be proven shortly.

We now prove (A.2) by assuming Lemma A.4.

Proof of (A.2). By the local wellposedness of the equation it is easy to see that the function K is smooth.

To prove the rest we apply Lemma A.4 by defining a new smooth function $u : \mathbb{R}^+ \rightarrow \mathbb{R}$ with the properties

$$u_t = K(t) \quad \text{if } t \leq T, \quad \text{and} \quad |u_t| \leq C(1+t)^{-2}$$

for some constant C . This function is well defined, because, in any finite time interval $[0, T]$, the solution K to the linear equation (4.1) exists. By direct computation we find that u satisfies the equation

$$\partial_t u_t = -Z \operatorname{Re} \langle W, \int_0^t e^{i \frac{\Delta}{2}(t-s)} u_s ds W \rangle - f_t, \quad (\text{A.23})$$

where the function f is defined by

$$f_t := -Z \operatorname{Re} \langle W, \int_0^t e^{i \frac{\Delta}{2}(t-s)} u_s ds W \rangle - \partial_t u_t : \mathbb{R}^+ \rightarrow \mathbb{R}$$

and satisfies the estimates

$$f_t = 0 \quad \text{if } t \leq T, \quad \text{and} \quad |f_t| \leq C_u(1+t)^{-\frac{3}{2}}.$$

These estimates are obtained easily, hence we omit the details.

Applying Lemma A.4 to (A.23) and using the fact that $f_t = 0$ if $t \leq T$, we find that if $t \leq T$ then $u_t = K(t)$ takes the desired form. Since T is arbitrary, this equation holds for any time. \square

Proof of Lemma A.4. We Fourier-transform both sides of (A.21) to derive an expression for $\int_0^\infty e^{ikt} \tilde{q}_t dt$:

$$\begin{aligned} \int_0^\infty e^{ikt} \partial_t \tilde{q}_t dt &= -\tilde{q}_0 - ik \int_0^\infty e^{ikt} \tilde{q}_t dt \\ &= -Z \int_0^\infty e^{ikt} \operatorname{Re} \langle W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} \tilde{q}_s ds W \rangle dt + \int_0^\infty e^{ikt} h(t) dt. \end{aligned}$$

An important observation is that the first term on the right-hand side admits a simpler expression:

$$- \int_0^\infty e^{ikt} \operatorname{Re} \langle W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} \tilde{q}_s ds W \rangle dt = G(k + i0) \int_0^\infty e^{ikt} \tilde{q}_t dt, \quad (\text{A.24})$$

where the function $G(\cdot + i0) : \mathbb{R} \rightarrow \mathbb{C}$ is defined in (A.1).

Collecting the identities above we find that

$$\int_0^\infty e^{ikt} \tilde{q}_t dt = -\frac{1}{ik + ZG(k + i0)} q_0 - \frac{1}{ik + ZG(k + i0)} \int_0^\infty e^{ikt} h(t) dt.$$

Inverse-Fourier transform of both sides of the equation yields

$$\tilde{q}_t = -(2\pi)^{-1} \int_{-\infty}^\infty \frac{1}{ik + ZG(k + i0)} e^{-ikt} dk \tilde{q}_0 + \frac{1}{2\pi} \int_{-\infty}^\infty e^{-ikt} H(k) dk,$$

where $H(k)$ is defined as

$$H(k) := -\frac{1}{ik + ZG(k + i0)} \int_0^\infty e^{ikt} h(t) dt.$$

By applying Fourier transformation, using (A.10) and the fact that $K(t) = 0$, for $t < 0$, we obtain that

$$\frac{1}{2\pi} \int_{-\infty}^\infty e^{-ikt} H(k) dk = \int_0^t K(t-s) h(s) ds. \quad (\text{A.25})$$

Collecting the estimates above we complete the proof. \square

A.4. Some decay estimates. In this part, we collect some estimates used in the main part of the paper.

Lemma A.5. *If W is a smooth, spherically symmetric and rapidly decaying function satisfying the condition $|\hat{W}(0)| = 1$ then, as $t \rightarrow \infty$,*

$$\begin{aligned} \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle &= -2t^{-\frac{3}{2}} \pi^{\frac{3}{2}} + O(t^{-\frac{5}{2}}), \\ 2\operatorname{Re} \langle W, (i\Delta)^{-1} e^{i\frac{\Delta}{2}t} W \rangle &= 4t^{-\frac{1}{2}} \pi^{\frac{3}{2}} + O(t^{-\frac{3}{2}}). \end{aligned} \quad (\text{A.26})$$

Proof. We start with proving the first equation. By Fourier transformation, introducing polar coordinates, and integrating by parts we find that

$$\begin{aligned}
 Re\langle W, e^{i\frac{\Delta}{2}t} W \rangle &= Re\langle \hat{W}, e^{-i\frac{|k|^2}{2}t} \hat{W} \rangle \\
 &= 4\pi \int_0^\infty \rho^2 \cos(\frac{1}{2}\rho^2 t) |\hat{W}(\rho)|^2 d\rho \\
 &= -\frac{4\pi}{t} \int_0^\infty \sin(\frac{1}{2}\rho^2 t) |\hat{W}(\rho)|^2 d\rho \\
 &\quad - \frac{4\pi}{t} \int_0^\infty \sin(\frac{1}{2}\rho^2 t) \rho \partial_\rho [|\hat{W}(\rho)|^2] d\rho \\
 &= -\frac{4\pi}{t} \left[\int_0^\infty \sin(\frac{1}{2}\rho^2 t) d\rho + \int_0^\infty \sin(\frac{1}{2}\rho^2 t) F(\rho) d\rho \right], \quad (\text{A.27})
 \end{aligned}$$

where $F(\rho)$ is defined by

$$F(\rho) := |\hat{W}(\rho)|^2 - 1 + \rho \partial_\rho |\hat{W}(\rho)|^2.$$

By rescaling $\rho(\frac{t}{2})^{\frac{1}{2}} \rightarrow \rho$ it is easy to see that the first term on the right-hand side takes the form

$$-\frac{4\pi}{t} \int_0^\infty \sin(\frac{1}{2}\rho^2 t) d\rho = -4\sqrt{2}\pi t^{-\frac{3}{2}} \int_0^\infty \sin(\rho^2) d\rho = -2\pi^{\frac{3}{2}} t^{-\frac{3}{2}}. \quad (\text{A.28})$$

Here the fact that $\int_0^\infty \sin x^2 dx = (\frac{\pi}{8})^{\frac{1}{2}}$ is used; (see (A.30)).

Now we turn to the second term. Define a new function, $\tilde{F}(\rho)$, by $\tilde{F}(\rho) := \rho^{-2} F(\rho)$. This changes our expression to

$$\int_0^\infty \sin(\frac{\rho^2 t}{2}) F(\rho) d\rho = \int_0^\infty \rho^2 \sin(\frac{\rho^2 t}{2}) \tilde{F}(\rho) d\rho.$$

Thanks to the fact that $F(\rho) = O(\rho^2)$, in a neighborhood of $\rho = 0$ and because of smoothness, we obtain that \tilde{F} is a smooth function. By standard techniques it is seen that

$$\int_0^\infty \sin(\frac{\rho^2 t}{2}) F(\rho) d\rho = O(t^{-\frac{3}{2}}). \quad (\text{A.29})$$

This together with (A.28) implies the first estimate in (A.26).

By similar arguments we obtain the second estimate in (A.26). \square

The following identity has been used in the proof:

$$\int_0^\infty \cos(x^2) dx = \int_0^\infty \sin(x^2) dx = (\frac{\pi}{8})^{\frac{1}{2}}. \quad (\text{A.30})$$

Proof. The key observation is that the function $e^{iz^2} = \cos z^2 + i \sin z^2 : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function. This together with the fact that in the region $Re z, Im z > 0$ the function decays rapidly, at $|z| \rightarrow \infty$, enables us to use the method of contour integration to obtain

$$\int_0^\infty e^{iz^2} dz = \int_{z \in [\frac{1}{\sqrt{2}}x + i\frac{1}{\sqrt{2}}x] : x \in [0, \infty)} e^{iz^2} dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) \int_0^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{\sqrt{8}} + \frac{\sqrt{\pi}}{\sqrt{8}}i.$$

This obviously implies (A.30). \square

B. Proof of Proposition 4.4

In the following we prove the three estimates in Proposition 4.4, in three different subsections. In various places we have to consider separately two cases: $\delta < \frac{1}{2}$, where functions in the Banach spaces $B_{\delta,T}$, see (4.11), are not necessarily integrable in time on the interval $[T, \infty)$, and $\delta > \frac{1}{2}$, where functions in $B_{\delta,T}$ are integrable. In order not to clutter our arguments with too much details, we only consider the cases $\delta < \frac{1}{2}$. The analysis for $\delta > \frac{1}{2}$ is similar and is considered in [3], where a slightly harder problem is solved.

Recall the ideas we presented after Proposition 4.4. In what follows we carry out the ideas in details.

B.1. Proof of (4.27). It is easy to estimate the second factor in the definition of Γ_2 : Recall that we only consider the case $\delta < \frac{1}{2}$,

$$|\int_0^t (1 - \chi_T(s_1)) q_{s_1} ds_1| \lesssim (1+t)^{\frac{1}{2}-\delta} \|q\|_{\delta,T}. \quad (\text{B.1})$$

The crucial step is to prove that

$$|\int_0^t K(t-s) \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle ds| \lesssim (1+t)^{-\frac{3}{2}}. \quad (\text{B.2})$$

Estimates (B.1) and (B.2) obviously imply (4.27).

In what follows we prove (B.2). Using the fact that $K(t) = 0$, for $t < 0$, in (A.3), applying Fourier transformation, using (A.10) and applying inverse Fourier transformation we find that

$$I := \int_0^t K(t-s) Z \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle ds \quad (\text{B.3})$$

$$= \int_0^\infty K(t-s) Z \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle ds \quad (\text{B.4})$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{F(k)}{ik + ZG(k+i0)} e^{-ikt} dk, \quad (\text{B.5})$$

where $F(k)$ is defined by

$$\begin{aligned} F(k) &:= Z \int_0^\infty e^{iks} \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle ds \\ &= \frac{Z}{2} \int_0^\infty e^{iks} [\langle W, e^{i\frac{\Delta}{2}s} W \rangle + \langle W, e^{-i\frac{\Delta}{2}s} W \rangle] ds \\ &= -Z[(\langle (-i\Delta + 2ik - 0)^{-1} W, W \rangle + \langle (i\Delta + 2ik - 0)^{-1} W, W \rangle)] \\ &= -ZG(k+i0), \end{aligned}$$

and $G(k+i0)$ is defined in (A.1).

By (A.5) the function $\frac{ZG(k+i0)}{ik+ZG(k+i0)}$ is smooth on $\mathbb{R} \setminus \{0\}$, and around $k = 0$ it has the expression

$$\frac{ZG(k+i0)}{ik+ZG(k+i0)} = -1 + C \frac{1}{Z} k^{\frac{1}{2}} + O(k)$$

for some constant C . Putting this into (B.3), and integrating by parts in the variable k , we obtain

$$\begin{aligned} |I| &= \frac{1}{2\pi} t^{-1} \left| \int_{-\infty}^{\infty} e^{-ikt} \partial_k \left[\frac{ZG(k+i0)}{ik + ZG(k+i0)} \right] dk \right| \\ &\leq \frac{1}{2\pi} t^{-1} [A_1 + A_2 + A_3] \end{aligned} \quad (\text{B.6})$$

with

$$\begin{aligned} A_1 &:= \left| \int_{-\infty}^{\infty} e^{-ikt} \frac{1}{ik + ZG(k+i0)} dk \right|, \\ A_2 &:= \left| \int_{-\infty}^{\infty} e^{-ikt} \frac{k}{[ik + ZG(k+i0)]^2} dk \right|, \\ A_3 &:= \left| \int_{-\infty}^{\infty} e^{-ikt} \frac{Zk \partial_k G(k+i0)}{[ik + ZG(k+i0)]^2} dk \right|. \end{aligned}$$

It is easy to see that $\frac{1}{ik+ZG(k+i0)} = O(|k|^{-\frac{1}{2}})$, $\frac{k}{[ik+ZG(k+i0)]^2} = O(1)$ and $\frac{Zk \partial_k G(k+i0)}{[ik+ZG(k+i0)]^2} = O(|k|^{-\frac{1}{2}})$, in a neighborhood of $k = 0$. This implies that

$$\sum_{k=1}^3 A_k = O(t^{-\frac{1}{2}}), \text{ or equivalently, } I = O(t^{-\frac{3}{2}}), \quad (\text{B.7})$$

which, together with the estimates above, yields the desired estimate (B.2). \square

B.2. Proof of (4.25). Recall the asymptotic forms of $K(t)$ and $\text{Re}\langle W, e^{i\frac{\Delta}{2}t} W \rangle$ in (4.2) and (A.26), respectively. We define functions \tilde{K} , \tilde{M} and $\tilde{\Gamma}_1$ to approximate these functions and Γ_1 :

$$\begin{aligned} Z\tilde{K}(t) &:= t^{-\frac{1}{2}} \frac{1}{2} \pi^{-\frac{5}{2}}, \\ \tilde{M}(t) &:= -2t^{-\frac{3}{2}} \pi^{\frac{3}{2}} \end{aligned} \quad (\text{B.8})$$

and

$$\tilde{\Gamma}_1(t) := -Z \int_0^t [\tilde{K}(t-s) - \tilde{K}(t)] \tilde{M}(s) \int_s^t q_{s_1} [1 - \chi_T(s_1)] ds_1 ds. \quad (\text{B.9})$$

In the following we prove a sharp estimate on $\tilde{\Gamma}_1$ and prove that $\Gamma_1 - \tilde{\Gamma}_1$ is negligibly small because it decays faster. Recall the definition of $\Omega_1(\delta)$ in (4.24), and recall that we only consider the case $\delta < \frac{1}{2}$.

Lemma B.1. *The function $\tilde{\Gamma}_1$ is estimated by*

$$|\tilde{\Gamma}_1|(t) \leq t^{-\frac{1}{2}-\delta} \Omega_1(\delta) \|q_t\|_{\delta, T}, \quad (\text{B.10})$$

and there exists a constant $\epsilon(T)$, with $\lim_{T \rightarrow \infty} \epsilon(T) = 0$ such that

$$|\tilde{\Gamma}_1 - \Gamma_1|(t) \leq t^{-\frac{1}{2}-\delta} \epsilon(T) \|q_t\|_{\delta, T}. \quad (\text{B.11})$$

Proof. We first prove (B.10). By direct computation we find

$$\begin{aligned}
 |\tilde{\Gamma}_1| &\leq \frac{1}{2\pi} \int_0^t [(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}] s^{-\frac{3}{2}} \int_s^t |q_{s_1}| ds_1 ds \\
 &\leq \frac{1}{(1-2\delta)\pi} \int_0^t [(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}] s^{-\frac{3}{2}} (t^{\frac{1}{2}-\delta} - s^{\frac{1}{2}-\delta}) ds \|q_t\|_{\delta,T} \\
 &= \frac{1}{(1-2\delta)\pi} \int_0^t (t-s)^{-\frac{1}{2}} t^{-\frac{1}{2}} \frac{1}{(t-s)^{\frac{1}{2}} + t^{\frac{1}{2}}} s^{-\frac{1}{2}} (t^{\frac{1}{2}-\delta} - s^{\frac{1}{2}-\delta}) ds \|q_t\|_{\delta,T}.
 \end{aligned}$$

Here $(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}$ has been rewritten as

$$(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}} = (t-s)^{-\frac{1}{2}} t^{-\frac{1}{2}} \frac{s}{(t-s)^{\frac{1}{2}} + t^{\frac{1}{2}}}.$$

Changing variable, $s = tr$, we see that

$$|\tilde{\Gamma}_1| \leq t^{-\frac{1}{2}-\delta} \Omega_1(\delta) \|q_t\|_{\delta,T}.$$

We proceed to estimating $\Gamma_1 - \tilde{\Gamma}_1$ and prove (B.11). We divide $\Gamma_1 - \tilde{\Gamma}_1$ into three parts, according to the integration regions:

$$\Gamma_1 - \tilde{\Gamma}_1 = I_1 + I_2 + I_3, \quad (\text{B.12})$$

with I_k , $k = 1, 2, 3$, defined by

$$\begin{aligned}
 I_1 &:= Z \int_0^{T^{\frac{1}{3}}} [K(t-s) - K(t)] \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^t (1 - \chi_T(s_1)) q_{s_1} ds_1 ds \\
 &\quad - Z \int_0^{T^{\frac{1}{3}}} [\tilde{K}(t-s) - \tilde{K}(t)] \tilde{M}(s) \int_s^t (1 - \chi_T(s_1)) q_{s_1} ds_1 ds, \\
 I_2 &:= Z \int_{T^{\frac{1}{3}}}^{t-T^{\frac{1}{3}}} [K(t-s) - K(t)] \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^t (1 - \chi_T(s_1)) q_{s_1} ds_1 ds \\
 &\quad - Z \int_{T^{\frac{1}{3}}}^{t-T^{\frac{1}{3}}} [\tilde{K}(t-s) - \tilde{K}(t)] \tilde{M}(s) \int_s^t (1 - \chi_T(s_1)) q_{s_1} ds_1 ds \\
 &= Z \int_{T^{\frac{1}{3}}}^{t-T^{\frac{1}{3}}} [(K(t-s) - K(t)) - (\tilde{K}(t-s) - \tilde{K}(t))] \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \\
 &\quad \times \int_s^t (1 - \chi_T(s_1)) q_{s_1} ds_1 ds + Z \int_{T^{\frac{1}{3}}}^{t-T^{\frac{1}{3}}} [\tilde{K}(t-s) - \tilde{K}(t)] [\text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle - \tilde{M}(s)] \\
 &\quad \times \int_s^t (1 - \chi_T(s_1)) q_{s_1} ds_1 ds,
 \end{aligned}$$

and

$$\begin{aligned}
 I_3 &:= Z \int_{t-T^{\frac{1}{3}}}^t [K(t-s) - K(t)] \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^t (1 - \chi_T(s_1)) q_{s_1} ds_1 ds \\
 &\quad - Z \int_{t-T^{\frac{1}{3}}}^t [\tilde{K}(t-s) - \tilde{K}(t)] \tilde{M}(s) \int_s^t (1 - \chi_T(s_1)) q_{s_1} ds_1 ds.
 \end{aligned}$$

We first analyze I_1 . When $s \leq T^{\frac{1}{3}}$ and $t \geq T$ we use (4.2) to obtain

$$\begin{aligned} & |K(t-s) - K(t)|, |\tilde{K}(t-s) - \tilde{K}(t)| \\ & \lesssim t^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}} \frac{s}{t^{\frac{1}{2}} + (t-s)^{\frac{1}{2}}} + [(t-s)^{-1} - t^{-1}] + (t-s)^{-\frac{3}{2}} \\ & \lesssim t^{-\frac{3}{2}}(1+s). \end{aligned}$$

Consequently

$$|K(t-s) - K(t)| |Re\langle W, e^{i\frac{\Delta}{2}s} W \rangle| + |\tilde{K}(t-s) - \tilde{K}(t)| |\tilde{M}(t)| \lesssim t^{-\frac{3}{2}} s^{-\frac{1}{2}}.$$

Plugging this into the definition of I_1 , we obtain

$$|I_1| \lesssim t^{-1-\delta} \int_0^{T^{\frac{1}{3}}} s^{-\frac{1}{2}} ds \|q\|_{\delta,T} = t^{-1-\delta} T^{\frac{1}{6}} \|q\|_{\delta,T} \leq T^{-\frac{1}{3}} t^{-\frac{1}{2}-\delta} \|q\|_{\delta,T}. \quad (\text{B.13})$$

Now we turn to I_2 . In the region $[T^{\frac{1}{3}}, t - T^{\frac{1}{3}}]$ the functions $\tilde{K}(t)$ and $\tilde{M}(t)$ are good approximations of $K(t)$ and $Re\langle W, e^{i\frac{\Delta}{2}t} W \rangle$. Specifically

$$\begin{aligned} |K(t-s) - \tilde{K}(t-s)| & \lesssim (1+t-s)^{-1} \\ |K(t) - \tilde{K}(t)| & \lesssim (1+t)^{-1} \\ |Re\langle W, e^{i\frac{\Delta}{2}s} W \rangle - \tilde{M}(s)| & \lesssim s^{-\frac{5}{2}}. \end{aligned}$$

By direct computation,

$$I_2 \lesssim t^{-1-\delta} \|q\|_{\delta,T} \leq T^{-\frac{1}{6}} t^{-\frac{1}{2}-\delta} \|q\|_{\delta,T}. \quad (\text{B.14})$$

Concerning I_3 , it is easy to find that in the region $s \geq t - T^{\frac{1}{3}}$ and $t \geq T$, $\langle W, e^{i\frac{\Delta}{2}s} W \rangle = O(t^{-\frac{3}{2}})$. Hence

$$\begin{aligned} |I_3| & \lesssim \int_{t-T^{\frac{1}{3}}}^t (|t-s|^{-\frac{1}{2}} + t^{-\frac{1}{2}}) ds t^{-1-\delta} \|q\|_{\delta,T} \\ & \lesssim T^{\frac{1}{6}} t^{-1-\delta} \|q\|_{\delta,T} \\ & \leq T^{-\frac{1}{3}} t^{-\frac{1}{2}-\delta} \|q\|_{\delta,T}. \end{aligned} \quad (\text{B.15})$$

We complete the proof by collecting the estimates above and adopting an appropriate definition of $\epsilon(T)$. \square

B.3. Proof of (4.26). Similarly as for Γ_1 , we start with retrieving the ‘main’ contribution to Γ_3 . We define a new function \tilde{V} to approximate the function $2Re\langle W, (i\Delta)^{-1} e^{i\frac{\Delta}{2}t} W \rangle$ when t is large (see (A.26)):

$$\tilde{V} := 4t^{-\frac{1}{2}} \pi^{\frac{3}{2}}.$$

We then define an approximation, $\tilde{\Gamma}_3$, of Γ_3 by setting

$$\tilde{\Gamma}_3 := Z \tilde{K}(t) \int_0^t [\tilde{V}(t-s) - \tilde{V}(t)] (1 - \chi_T(s)) q_s ds, \quad (\text{B.16})$$

where, $\tilde{K}(t)$ has been defined in (B.8).

Recall the definitions of $\|q\|_{\delta,T}$ and $\Omega_2(\delta)$ in (4.12) and (4.24), respectively, and recall that we only consider $\delta < \frac{1}{2}$.

Proposition B.2. *There exists a time $T > 0$ such that, for any $t \geq T$,*

$$|\tilde{\Gamma}_3| \leq t^{-\frac{1}{2}-\delta} \Omega_2(\delta) \|q\|_{\delta,T}. \quad (\text{B.17})$$

The remainder $\Gamma_3 - \tilde{\Gamma}_3$ is estimated by

$$|\Gamma_3 - \tilde{\Gamma}_3| \leq t^{-\frac{1}{2}-\delta} \epsilon(T) \|q\|_{\delta,T}, \quad (\text{B.18})$$

where $\epsilon(T)$ is a small constant satisfying $\lim_{T \rightarrow \infty} \epsilon(T) = 0$.

Proof. We start with a proof of (B.17). By direct computation

$$\begin{aligned} |\tilde{\Gamma}_3| &\leq t^{-\frac{1}{2}} \frac{1}{\pi} \int_0^t [(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}] s^{-\frac{1}{2}-\delta} ds \|q\|_{\delta,T} \\ &= t^{-1} \frac{1}{\pi} \int_0^t \frac{s}{(t-s)^{\frac{1}{2}} + t^{\frac{1}{2}}} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}-\delta} ds \|q\|_{\delta,T}. \end{aligned}$$

Changing variables $s := tr$ we obtain the desired estimate

$$\begin{aligned} |\tilde{\Gamma}_3| &\leq t^{-\frac{1}{2}-\delta} \frac{1}{\pi} \int_0^1 (1-r)^{-\frac{1}{2}} \frac{1}{1+(1-r)^{\frac{1}{2}}} r^{\frac{1}{2}-\delta} dr \|q\|_{\delta,T} \\ &= t^{-\frac{1}{2}-\delta} \Omega_2(\delta) \|q\|_{\delta,T}. \end{aligned}$$

To prove (B.18) we transform $\Gamma_3 - \tilde{\Gamma}_3$ into a more convenient form:

$$\begin{aligned} \Gamma_3 - \tilde{\Gamma}_3 &= Z[K(t) - \tilde{K}(t)] \int_0^t [V(t-s) - V(t)] (1 - \chi_T(s)) q_s ds \\ &\quad + Z\tilde{K}(t) \int_0^t [V(t-s) - \tilde{V}(t-s)] (1 - \chi_T(s)) q_s ds \\ &\quad + Z\tilde{K}(t) [\tilde{V}(t) - V(t)] \int_0^t (1 - \chi_T(s)) q_s ds, \end{aligned} \quad (\text{B.19})$$

where the function $V(t)$ is defined by

$$V(t) := 2\text{Re} \langle W, (i\Delta)^{-1} e^{i\frac{\Delta}{2}t} W \rangle.$$

By arguments almost identical to the proof of (B.11) we conclude that (B.18) holds. \square

C. Proof of (4.19)

In this section we prove inequality (4.19) in Theorem 4.2. As in Appendix B, we only consider the case $\delta < \frac{1}{2}$. The case $\delta > \frac{1}{2}$ is considered in [3].

Proposition C.1. *Let Q_1 and Q_2 be defined in Theorem 4.2, and recall the definitions of $F(P)$ and B_k , $k = 0, 1, 2$, in (3.7).*

If $|P_0| \leq T^{-3}$, then

$$|B_0(Q_1) - B_0(Q_2)|(t) \ll (1+t)^{-1-\delta} \|Q_1 - Q_2\|_{\delta, T}, \quad (\text{C.1})$$

and, for $k = 1, 2$,

$$|B_k(Q_1) - B_k(Q_2)|(t) \lesssim (1+t)^{-1-\delta} \|Q_1 - Q_2\|_{\delta, T} (\|Q_1\|_{\delta, T} + \|Q_2\|_{\delta, T}). \quad (\text{C.2})$$

This proposition is proven below.

Now we use this result to prove the desired estimate (4.19).

Proof of (4.19). We start by estimating $K(t-s) - K(t)$. Using (4.2), we obtain

$$\begin{aligned} |K(t-s) - K(t)| &\lesssim |(1+t-s)^{-\frac{1}{2}} - (1+t)^{-\frac{1}{2}}| + (1+t-s)^{-1} - (1+t)^{-1} \\ &\quad + (1+t-s)^{-\frac{3}{2}} \\ &= (1+t-s)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}} \frac{s}{(1+t)^{\frac{1}{2}} + (1+t-s)^{\frac{1}{2}}} \\ &\quad + s(1+t-s)^{-1}(1+t)^{-1} + (1+t-s)^{-\frac{3}{2}} \\ &\leq (1+t-s)^{-\frac{1}{2}}(1+t)^{-1}s + (1+t-s)^{-\frac{3}{2}}. \end{aligned} \quad (\text{C.3})$$

Plugging this estimate, as well as (C.1) and (C.2) into $\int_0^t |K(t-s) - K(t)| |B_k(Q_1) - B_k(Q_2)|(s) ds$, $k = 0, 1, 2$, we obtain the desired estimate (4.19). \square

C.1. Proof of (C.1). By direct computation and our estimate on $\|e^{i\frac{\Delta}{2}t}\|_{L^1 \rightarrow L^\infty}$, we obtain that

$$\begin{aligned} |B_0(Q_1) - B_0(Q_2)|(t) &\lesssim t^{-\frac{3}{2}} \|\beta_0\|_{L^1} \|\partial_x W \int_T^t [Q_1(s) - Q_2(s)] ds - \partial_x W\|_{L^1} \\ &\lesssim t^{-\frac{3}{2}} \|\langle x \rangle^4 \beta_0\|_{L^2} \|\partial_x W \int_T^t [Q_1(s) - Q_2(s)] ds - \partial_x W\|_{L^1}. \end{aligned}$$

The right-hand side vanishes when $t \leq T$. This, together with $T \gg 1$, implies that

$$\begin{aligned} |B_0(Q_1) - B_0(Q_2)|(t) &= (1+t)^{-\frac{3}{2}} \|\langle x \rangle^4 \beta_0\|_{L^2} \left\| \int_T^t \partial_s \partial_x W \int_T^s [Q_1(s_1) - Q_2(s_1)] ds_1 ds \right\|_{L^1} \\ &\lesssim (1+t)^{-\frac{3}{2}} \|\langle x \rangle^4 \beta_0\|_{L^2} \|\partial_x W\|_{L^1} \int_T^t |Q_1(s) - Q_2(s)| ds \\ &\ll (1+t)^{-1-\delta} \|Q_1 - Q_2\|_{\delta, T}. \end{aligned}$$

Here the hypothesis of smallness of $\|\langle x \rangle^4 \beta_0\|_{L^2}$ in Theorem 2.1 is used, and we recall that only the case $\delta < \frac{1}{2}$ is considered. \square

C.2. Proof of (C.2). In what follows we only estimate the contributions to B_2 ; the estimate of B_1 is similar and easier, hence omitted.

We start with transforming B_2 . Using the formula

$$W^{X_s - X_t} - W = - \int_s^t \partial_{s_1} W^{X_{s_1} - X_t} ds_1 = \int_s^t [P_{s_1} \cdot \partial_x] W^{X_{s_1} - X_t} ds_1,$$

we obtain

$$\begin{aligned} B_2(P) &= -2\nu \operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} [P_s \cdot \partial_x] \int_s^t [P_{s_1} \cdot \partial_x] W^{X_{s_1} - X_t} ds_1 ds \rangle \\ &= -2\nu \operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} [P_s \cdot \partial_x] \\ &\quad \times \int_s^t [P_{s_1} \cdot \partial_x] [W^{X_{s_1} - X_t} - W] ds_1 ds \rangle \\ &= -2\nu \operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} [P_s \cdot \partial_x] \int_s^t [P_{s_1} \cdot \partial_x] \\ &\quad \times \int_{s_1}^t [P_{s_2} \cdot \partial_x] W^{X_{s_2} - X_t} ds_2 ds_1 ds \rangle. \end{aligned} \quad (\text{C.4})$$

Here the fact that the function W is spherically symmetric has been used to see that various terms vanish.

Next, we prove (C.2) for $k = 2$. Among many terms in its expression we only consider

$$\begin{aligned} \tilde{B} &:= -2\nu \operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} [Q_1(s) \cdot \partial_x] \int_s^t [Q_1(s_1) \cdot \partial_x] \\ &\quad \times \int_{s_1}^t [Q_1(s_2) \cdot \partial_x] [W^{X_{s_2} - X_t} - W^{\tilde{X}_{s_2} - \tilde{X}_t}] ds_2 ds_1 ds \rangle, \end{aligned}$$

where $X_t := X_0 + \int_0^t Q_1 s$ and $\tilde{X}_t := X_0 + \int_0^t Q_2(s) ds$. We rewrite $W^{X_{s_2} - X_t} - W^{\tilde{X}_{s_2} - \tilde{X}_t}$ as follows:

$$\begin{aligned} W^{X_{s_2} - X_t} - W^{\tilde{X}_{s_2} - \tilde{X}_t} &= - \int_{s_2}^t \partial_z W^{X_z - X_t - \tilde{X}_z + \tilde{X}_{s_2}} dz \\ &= \int_{s_2}^t (Q_1(z) - Q_2(z)) \cdot \nabla_x W^{X_z - X_t - \tilde{X}_z + \tilde{X}_{s_2}} dz. \end{aligned}$$

Consequently

$$\begin{aligned} B_0 &= -2\nu \operatorname{Re} \langle \nabla_x W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} [Q_1(s) \cdot \partial_x] \int_s^t [Q_1(s_1) \cdot \partial_x] \\ &\quad \times \int_{s_1}^t [Q_1(s_2) \cdot \partial_x] \int_{s_2}^t (Q_1(z) - Q_2(z)) \cdot \nabla_x W^{X_z - X_t - \tilde{X}_z + \tilde{X}_{s_2}} dz ds_2 ds_1 ds \rangle. \end{aligned}$$

To obtain a decay estimate we have to consider a function I_4 defined by

$$\begin{aligned} I_4 &:= \langle \nabla_x W, e^{i\frac{\Delta}{2}(t-s)} (-\Delta)^{-1} \left(\prod_{k_l \in \{1,2,3\}, l=1,2,3} \partial_{x_{k_l}} \right) W^{Y(t,s_2,z)} \rangle \\ &= \langle \nabla_x W, e^{i\frac{\Delta}{2}(t-s) + Y(t,s_2,z) \cdot \partial_x} (-\Delta)^{-1} \left(\prod_{k_l \in \{1,2,3\}, l=1,2,3} \partial_{x_{k_l}} \right) W \rangle, \end{aligned}$$

where $Y(t, s_2, z) \in \mathbb{R}^3$ is defined by $Y(t, s_2, z) := X_z - X_t - \tilde{X}_z + \tilde{X}_{s_2}$, with $s_2 \in [s, t]$ and $z \in [s_2, t]$. It is easy to transform I_4 to a form where Proposition C.2, below, applies, which yields

$$|I_4| \lesssim (1+t-s)^{-3}, \quad \text{for any } t \text{ and } s \leq t. \quad (\text{C.5})$$

Now it is easy to see that, for some constant C ,

$$\begin{aligned} |\tilde{B}| &\lesssim \int_0^t (1+t-s)^{-3} s^{-\frac{1}{2}-\delta} \int_s^t s_1^{-\frac{1}{2}-\delta} \int_{s_1}^t s_2^{-\frac{1}{2}-\delta} \\ &\quad \times \int_{s_2}^t s_3^{-\frac{1}{2}-\delta} ds_3 ds_2 ds_1 ds \|Q_1\|_{\delta, T}^3 \|Q_1 - Q_2\|_{\delta, T} \end{aligned} \quad (\text{C.6})$$

$$= C[t^{\frac{3}{2}-3\delta} \int_0^t (1+t-s)^{-3} s^{-\frac{1}{2}-\delta} ds - 3t^{1-2\delta} \int_0^t (1+t-s)^{-3} s^{-2\delta} ds \quad (\text{C.7})$$

$$+ 3t^{\frac{1}{2}-\delta} \int_0^t (1+t-s)^{-3} s^{\frac{1}{2}-3\delta} ds - \int_0^t (1+t-s)^{-3} s^{1-4\delta} ds] \quad (\text{C.8})$$

$$\times \|Q_1\|_{\delta, T}^3 \|Q_1 - Q_2\|_{\delta, T}. \quad (\text{C.9})$$

To estimate the integrals above we divide the interval $[0, t]$ into two subintervals: $[0, \frac{1}{2}t]$ and $[\frac{1}{2}t, t]$, and denote the corresponding contributions by D_1 and D_2 respectively.

(1) Concerning D_1 , it is easy to see that

$$D_1 \lesssim t^{-1-4\delta} \|Q_1\|_{\delta, T}^3 \|Q_1 - Q_2\|_{\delta, T},$$

using the fact that $(1+t-s)^{-3} = O((1+t)^{-3})$ and direct computation.

(2) In the second interval we Taylor-expand the functions $s^\alpha = t^\alpha [1 - \alpha \frac{t-s}{t} + O([\frac{t-s}{t}]^2)]$, with $\alpha = -\frac{1}{2} - \delta$, -2δ , $\frac{1}{2} - 3\delta$, and find that the leading terms cancel each other. This implies that

$$\begin{aligned} D_2 &\lesssim t^{-1-4\delta} \int_{\frac{1}{2}t}^t (1+t-s)^{-1} ds \|Q_1\|_{\delta, T}^3 \|Q_1 - Q_2\|_{\delta, T} \\ &\lesssim t^{-1-4\delta} \ln t \|Q_1\|_{\delta, T}^3 \|Q_1 - Q_2\|_{\delta, T}. \end{aligned}$$

Collecting the above estimates we conclude that

$$\tilde{B}(t) \lesssim t^{-1-2\delta} \|Q_1\|_{\delta, T}^3 \|Q_1 - Q_2\|_{\delta, T}. \quad (\text{C.10})$$

To remove the non-integrable singularity at $t = 0$, we use the fact that $\tilde{B}(t) = 0$, for $t < T$, with $T \gg 1$. \square

C.3. *Some decay estimates.* The following result has been used in (C.5).

Proposition C.2. *Suppose that the functions $f : \mathbb{R}^+ \rightarrow \mathbb{C}$, $h : [0, 2\pi] \rightarrow \mathbb{C}$ and $y : \mathbb{R}^+ \rightarrow \mathbb{R}$ are smooth and satisfy the conditions:*

$$\begin{aligned} |y(t)|t^{-\frac{1}{2}} &\leq \epsilon_0, \quad \text{for some small constant } \epsilon_0 > 0; \\ |\partial_\rho^k f(\rho)| &\leq a_k e^{-c_0|\rho|}, \quad \text{for any } k \in \mathbb{N}, \text{ where } a_k, c_0 > 0 \text{ are constants.} \end{aligned}$$

Then there exist constants C_l , $l \in \mathbb{N}$, such that

$$|\int_0^{2\pi} h(\theta) \int_0^\infty e^{i\rho^2 t} e^{i\rho \cos \theta y(t)} \rho^l f(\rho^2) d\rho d\theta| \leq C_l (1+t)^{-\frac{l+1}{2}}. \quad (\text{C.11})$$

Proof. If $y \equiv 0$ the proof is standard, (integration by parts). In the present situation we use integration by parts and the fact that $y(t)$ is appropriately small.

For notational purposes, we define functions I_l by

$$I_l(t) := \int_0^{2\pi} h(\theta) \int_0^\infty e^{i\rho^2 t} e^{i\rho \cos \theta y(t)} \rho^l f(\rho^2) d\rho d\theta.$$

We prove by induction that

$$|I_l| \leq C_l (1+t)^{-\frac{l+1}{2}}.$$

Step 1. For $l = 0$, we change variables, $\rho^2 t =: r$, to obtain

$$\begin{aligned} I_0 &= \frac{1}{2} t^{-\frac{1}{2}} \int_0^{2\pi} h(\theta) \int_0^\infty r^{-\frac{1}{2}} e^{ir} e^{ir^{\frac{1}{2}} \cos \theta y(t) t^{-\frac{1}{2}}} f(t^{-1}r) dr d\theta \\ &= \frac{1}{2} t^{-\frac{1}{2}} [J_1 + J_2] \end{aligned} \quad (\text{C.12})$$

with

$$J_1 := \int_0^{2\pi} h(\theta) \int_0^1 r^{-\frac{1}{2}} e^{ir} e^{ir^{\frac{1}{2}} \cos \theta y(t) t^{-\frac{1}{2}}} f(t^{-1}r) dr d\theta$$

and

$$J_2 := \int_0^{2\pi} h(\theta) \int_1^\infty r^{-\frac{1}{2}} e^{ir} e^{ir^{\frac{1}{2}} \cos \theta y(t) t^{-\frac{1}{2}}} f(t^{-1}r) dr d\theta.$$

J_1 is clearly bounded.

To bound J_2 an obvious obstacle is that the function $r^{-\frac{1}{2}} \notin L^1[0, \infty)$. The way out is to integrate by parts, using that

$$-i \frac{1}{1 + \frac{1}{2} r^{-\frac{1}{2}} \cos \theta y(t) t^{-\frac{1}{2}}} \partial_r [e^{ir} e^{ir^{\frac{1}{2}} \cos \theta y(t) t^{-\frac{1}{2}}}] = e^{ir} e^{ir^{\frac{1}{2}} \cos \theta y(t) t^{-\frac{1}{2}}},$$

and to use that the function

$$g(r, \theta) := -i \frac{1}{1 + \frac{1}{2} r^{-\frac{1}{2}} \cos \theta y(t) t^{-\frac{1}{2}}}$$

is uniformly bounded, for $r \geq 1$, as follows from the hypotheses of the proposition. Then

$$|J_2| \leq \int_0^{2\pi} |h(\theta)| |g(1, \theta)| |f(t^{-1})| d\theta \\ + \int_0^{2\pi} |h(\theta)| \int_1^\infty |\partial_r [r^{-\frac{1}{2}} g(r, \theta) f(t^{-1}r)]| dr d\theta.$$

The first term on the right-hand side is obviously bounded. For the second term, it is not difficult to derive that

$$|\partial_r [r^{-\frac{1}{2}} g(r, \theta) f(t^{-1}r)]| \lesssim r^{-\frac{3}{2}} + t^{-1} r^{-\frac{1}{2}} |f'(t^{-1}r)|.$$

Using our assumptions on the function f we find that

$$t^{-1} \int_1^\infty r^{-\frac{1}{2}} |f'(t^{-1}r)| dr \lesssim t^{-\frac{1}{2}}.$$

Collecting the estimates above, $|J_2|$ is seen to be bounded. This, together with (C.12) and the fact J_1 is bounded, implies the desired result.

Step 2. In the second step of the induction we assume that (C.11) holds for any $k \leq l$.

Now we estimate I_{l+1} . The idea is to represent I_{l+1} into a linear combination of I_{l-1} and I_l by performing integration by parts on certain variables; (with $I_{-1} = 0$).

Integrating by parts, and using the identity $-\frac{i}{2}t^{-1}\partial_\rho e^{i\rho^2 t} = \rho e^{i\rho^2 t}$, we find that

$$I_{l+1} = it^{-1} \int_0^\pi h(\theta) \int_0^{2\infty} e^{i\rho^2 t} \partial_\rho [e^{i\rho \cos \theta} y(t) \rho^l f(\rho^2)] d\rho d\theta \\ = t^{-1} y(t) K_1 + t^{-1} K_2$$

with

$$K_1 := -\frac{1}{2} \int_0^{2\pi} \cos \theta h(\theta) \int_0^\infty e^{i\rho^2 t} e^{i\rho \cos \theta} y(t) \rho^l f(\rho^2) d\rho d\theta$$

and

$$K_2 := \frac{i}{2} \int_0^{2\pi} h(\theta) \int_0^\infty e^{i\rho^2 t} e^{i\rho \cos \theta} y(t) \partial_l [\rho^l f(\rho^2)] d\rho d\theta.$$

Notice K_1 and K_2 are of the form of I_l or I_{l-1} (if $l \geq 1$), after defining appropriate functions $f_{\text{new}}(\rho^2)$ and $h_{\text{new}}(\theta)$. Hence the induction hypotheses on $I_k, k \leq l$ imply that

$$|I_{l+1}| \lesssim |t^{-1} y(t)| t^{-\frac{l+1}{2}} + t^{-\frac{l+2}{2}} \lesssim t^{-\frac{l+2}{2}},$$

which is the desired estimate on I_{l+1} .

Thus, (C.11) holds for any $l \in \mathbb{Z}^+$. \square

D. Proof of Theorem 4.3

We reformulate this theorem in the form of the following lemma. Recall the constant $\epsilon_0(T)$ in Theorem 3.1 and the definition of $F_{\chi_T P}$ in (3.7).

Lemma D.1. *If T is sufficiently large and $|P_0|, \|\langle x \rangle^4 \beta_0\|_2 \leq \epsilon_0(T)$ then, for any time $t \geq T$,*

$$|A(\chi_T P)| \leq \epsilon(T)(1+t)^{-\frac{1}{2}-\delta}; \quad (\text{D.1})$$

$$|\int_0^t (K(t-s) - K(t)) B_0(\chi_T P)(s) ds| \lesssim \epsilon(T)(1+t)^{-\frac{3}{2}} \|\langle x \rangle^4 \beta_0\|_2; \quad (\text{D.2})$$

and, for $k = 1, 2$,

$$|\int_0^t (K(t-s) - K(t)) B_k(\chi_T P)(s) ds| \leq \epsilon(T)t^{-1-2\delta}, \quad (\text{D.3})$$

where $\epsilon(T)$ is a small constant satisfying $\lim_{T \rightarrow \infty} \epsilon(T) = 0$.

Proof. We start with proving (D.1). By its definition in (4.10), the function $A(\chi_T P)$ takes the form

$$\begin{aligned} A(\chi_T P) = & -Z \int_0^t [K(t-s) - K(t)] \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^t P_{s_1} \chi_T(s_1) ds_1 ds \\ & + Z \int_0^t K(t-s) \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle ds \int_0^t P_{s_1} \chi_T(s_1) ds_1 \\ & + 2Z K(t) \text{Re}\langle W, (i\Delta)^{-1} \int_0^t [e^{i\frac{\Delta}{2}(t-s)} - e^{i\frac{\Delta}{2}t}] P_s \chi_T(s) ds W \rangle. \end{aligned}$$

Using (3.10) we obtain

$$\begin{aligned} |A(\chi_T P)| \leq & Z T^{-1} [\int_0^T |K(t-s) - K(t)| |\text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle| ds \\ & + |\int_0^t K(t-s) \text{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle ds| \\ & + |K(t)| \int_0^T |\text{Re}\langle W, (i\Delta)^{-1} [e^{i\frac{\Delta}{2}(t-s)} - e^{i\frac{\Delta}{2}t}] W \rangle| ds]. \quad (\text{D.4}) \end{aligned}$$

As proven in (B.2), the second term on the right-hand side is of order $t^{-\frac{3}{2}}$.

We now turn to the remaining two terms. For $\text{Re}\langle W, (i\Delta)^{-1} [e^{i\frac{\Delta}{2}(t-s)} - e^{i\frac{\Delta}{2}t}] W \rangle$, in the last line, we use estimates similar to (C.3) to conclude that

$$|\text{Re}\langle W, (i\Delta)^{-1} [e^{i\frac{\Delta}{2}(t-s)} - e^{i\frac{\Delta}{2}t}] W \rangle| \lesssim (1+t-s)^{-\frac{1}{2}} (1+t)^{-1} s + (1+t-s)^{-1}.$$

Putting this and (C.3) back into (D.4), we obtain

$$\begin{aligned} |A(\chi_T P)| \leq & T^{-1} [(1+t)^{-1} \int_0^T (1+t-s)^{-\frac{1}{2}} \frac{s}{(1+s)^{\frac{3}{2}}} ds + \int_0^t (1+t-s)^{-1} (1+s)^{-\frac{3}{2}} ds \\ & + (1+t)^{-\frac{3}{2}} \int_0^T (1+t-s)^{-\frac{1}{2}} s ds + (1+t)^{-\frac{1}{2}} \int_0^T (1+t-s)^{-1} ds] \\ \leq & T^{-\frac{1}{3}} (1+t)^{-\frac{1}{2}-\delta}, \end{aligned}$$

where, to obtain the last line, we consider two regimes, $t \in [T, 2T]$ and $t \geq 2T$.

The proof of (D.1) is completed by setting $\epsilon(T) = T^{-\frac{1}{3}}$.

To prove (D.2) we use standard arguments to conclude that

$$|B_0(\chi_T P)| \lesssim \|\langle x \rangle^{-4} e^{-i\Delta t} \nabla_x W^{\chi_T P}\|_2 \|\langle x \rangle^4 \beta_0\|_2 \lesssim (1+t)^{-2} \|\langle x \rangle^4 \beta_0\|_2.$$

This, together with (C.3) and the fact $t \geq T$, implies (D.2).

To prove (D.3) we only estimate B_1 , the estimate on B_2 being very similar.

Recall the definition of P_t . Applying the formula

$$W^{X_0-X_s} - W = \int_0^s \partial_{s_1} W^{X_0-X_{s_1}} ds_1 = \int_0^s [P_{s_1} \cdot \partial_x] W^{X_0-X_{s_1}} ds_1,$$

we rewrite the expression for B_1 :

$$B_1(P) = 2\nu Re \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} \int_0^t [P_s \cdot \partial_x] \int_0^s [P_{s_1} \cdot \partial_x] W^{X_0-X_{s_1}} ds_1 ds \rangle.$$

Define

$$\tilde{X}_t := X_0 + \int_0^t \chi_T P_s ds.$$

Then

$$\begin{aligned} B_1(\chi_T P) \\ = 2\nu Re \langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} \int_0^t [P_s \cdot \partial_x] \int_0^s [P_{s_1} \cdot \partial_x] W^{X_0-\tilde{X}_{s_1}} ds_1 ds \rangle. \end{aligned}$$

By the estimate on $\chi_T P_t$ in (3.10), it is easy to see that

$$\int_0^t \chi_T |P_s| ds = \int_0^T \chi_T |P_s| ds \leq T^{-1}. \quad (\text{D.5})$$

By a similar argument as in (C.5), we may apply Proposition C.2 to find that

$$|\langle \nabla_x W, e^{i\frac{\Delta}{2}t} (-\Delta)^{-1} \partial_{x_k} \partial_{x_l} W^{X_0-\tilde{X}_{s_1}} \rangle| \lesssim (1+t)^{-2},$$

for any $s_1 \in [0, t]$. This, together with (3.10), implies that

$$|B_1(\chi_T P)| \lesssim (1+t)^{-2} T^{-2},$$

which, together with (C.3), implies (D.3). \square

E. Some Heuristic Ideas Underlying (4.9)

We emphasize that the non-rigorous discussions in this subsection are NOT used in any other parts of the paper.

In this section we present the ideas underlying Eq. (4.9), which is the key equation in our proof of the most important result, Theorem 4.2.

To save space we only consider the linear part of (3.8), which corresponds to the following equation:

$$\begin{aligned} \dot{q}_t &= Z \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} t} W \rangle \int_0^t q_s ds - Z \operatorname{Re} \langle W, \int_0^t e^{i \frac{\Delta}{2} (t-s)} q_s ds W \rangle, \\ q_0 &= 1. \end{aligned} \quad (\text{E.1})$$

By repeating the arguments used to prove Theorems 4.2 and 4.3 we prove that the solution q_t belongs to the Banach space $B_{\delta, T}$.

We divide our discussion into two parts: In Subsect. E.1, we present the ideas behind constructing (4.9), assuming that $|q_t| \leq \text{const } t^{-\frac{1}{2}-\delta}$ for some $\delta > 0$. In Subsect. E.2, we present a heuristic argument to show that $|q_t| \leq \text{const } t^{-\frac{1}{2}-\delta}$, for some $\delta > 0$.

E.1. Ideas underlying (4.9). Recall that (4.9) is obtained by subtracting from (4.6) the product of (4.7) with the function $K(t)$.

We define a constant

$$C_0 := q_0 - 2Z \int_0^\infty \operatorname{Re} \langle W, (i\Delta)^{-1} e^{i \frac{\Delta}{2} s} W \rangle q_s ds.$$

Lemma E.1. *If the function q of (E.1) satisfies the estimate $|q_t| \leq ct^{-\frac{1}{2}-\delta}$, for some $\delta > 0$, then*

$$C_0 = 0. \quad (\text{E.2})$$

This lemma will be proven shortly.

For technical reasons, it is hard to prove that $|q_t| \leq Ct^{-\frac{1}{2}-\delta}$, even after imposing $C_0 = 0$ on (4.6). Instead we search for a new equation containing C_0 . A natural candidate is (4.7), which is obtained by integrating both sides of (3.8) from 0 to t . The key observation is:

Lemma E.2. *If the function q satisfies the estimate $|q_t| \leq ct^{-\frac{1}{2}-\delta}$, for some $\delta > 0$, then (4.7) can be rewritten as*

$$q_t = C_0 + O(t^{-\delta}). \quad (\text{E.3})$$

The proof of this lemma is almost identical to that of Lemma E.1.

By Lemmas E.1 and E.2 it is natural to consider Eq. (4.9), as we did.

Proof of Lemma E.1. The key point in proving the lemma is to show that

$$\begin{aligned} & \int_0^t K(t-s) \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle \int_0^s q_{s_1} ds_1 ds \\ &= K(t) \int_0^t \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle \int_0^s q_{s_1} ds_1 ds + O(t^{-\frac{1}{2}-\delta}). \end{aligned} \quad (\text{E.4})$$

Suppose this holds. Then by integrating by parts in the variable s and the assumption that $|q_t| \leq Ct^{-\frac{1}{2}-\delta}$, we find

$$\begin{aligned} & \int_0^t K(t-s) \operatorname{Re} \langle W, e^{i \frac{\Delta}{2} s} W \rangle \int_0^s q_{s_1} ds_1 ds \\ &= -2K(t) \int_0^t \operatorname{Re} \langle W, (i\Delta)^{-1} e^{i \frac{\Delta}{2} s} W \rangle q_s ds + O(t^{-\frac{1}{2}-\delta}) \\ &= -2K(t) \int_0^\infty \operatorname{Re} \langle W, (i\Delta)^{-1} e^{i \frac{\Delta}{2} s} W \rangle q_s ds + O(t^{-\frac{1}{2}-\delta}) \end{aligned}$$

which, together with (E.1) and the estimate $Re\langle W, (i\Delta)^{-1}e^{i\frac{\Delta}{2}t}W \rangle = O(t^{-\frac{1}{2}})$ proved in (A.26), obviously implies (E.2).

Now we turn to (E.4). Define a function $f : [0, \infty) \rightarrow \mathbb{R}$ by

$$f(t) := Re\langle W, e^{i\frac{\Delta}{2}t}W \rangle \int_0^t q_{s_1} ds_1.$$

Our assumption on q_t and the estimate $Re\langle W, e^{i\frac{\Delta}{2}t}W \rangle = O(t^{-\frac{3}{2}})$ in (A.26) imply that there exists a constant C_1 such that

$$|f(t)| \leq C_1(1+t)^{-1-\delta}.$$

To prove (E.4) it is sufficient to prove that

$$\int_0^t |K(t-s) - K(t)| |f(s)| ds = O(t^{-\frac{1}{2}-\delta}). \quad (\text{E.5})$$

To see this we use the asymptotics of the function K in (4.2) and obtain that

$$\begin{aligned} \int_0^t |K(t-s) - K(t)| |f(s)| ds &\lesssim \int_0^t [(t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}] (1+s)^{-1-\delta} ds \\ &\quad + \int_0^t [(1+t-s)^{-\frac{3}{2}} + (1+t)^{-\frac{3}{2}}] (1+s)^{-1-\delta} ds. \end{aligned}$$

The second term on the right-hand side is of order $t^{-1-\delta}$. For the first term, we use the observation that

$$0 < (t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}} = (t-s)^{-\frac{1}{2}} t^{-\frac{1}{2}} \frac{s}{(t-s)^{\frac{1}{2}} + t^{\frac{1}{2}}} \leq t^{-1} (t-s)^{-\frac{1}{2}} s.$$

A direct computation then shows that it is of order $t^{-\frac{1}{2}-\delta}$.

Hence (E.5) is proved. \square

E.2. Best decay estimate. Here we consider the decay estimate for the solution of the linear integro-differential equation (E.1). Recall that Z is a positive constant, $W : \mathbb{R}^3 \rightarrow \mathbb{R}$ is spherically symmetric and of rapid decay.

We consider two possibilities: The solution q decays faster and slower than t^{-1} , respectively. In the first case we have the following result.

Theorem E.3. *Suppose that the solution $q : \mathbb{R}^+ \rightarrow \mathbb{R}$ of (E.1) satisfies the asymptotic form*

$$q_t = Ct^{-1-\delta} + O(t^{-1-\delta-\epsilon}), \quad \text{as } t \rightarrow \infty, \quad (\text{E.6})$$

for some $\delta \in (0, \frac{1}{2})$ and $\epsilon > 0$. Then the constant δ must satisfy the equation

$$\int_0^\infty e^{-t} t^{-\delta} dt = \frac{1}{\sqrt{\pi}(1+2\delta)} \int_0^\infty e^{-t} t^{-\frac{1}{2}-\delta} dt. \quad (\text{E.7})$$

Remark 3. Concerning (E.7), by computer simulation there exists exactly one solution δ , which belongs to the interval $(0.15, 0.17)$.

In the second case we have the following result.

Theorem E.4. *There does not exist $\delta \in (0, \frac{1}{2})$ such that the solution $q : \mathbb{R}^+ \rightarrow \mathbb{R}$ of Eq. (E.1) has the asymptotic form*

$$q_t = Ct^{-\frac{1}{2}-\delta} + O(t^{-\frac{1}{2}-\delta-\epsilon}), \quad (\text{E.8})$$

for some $\epsilon > 0$. Equivalently, there is no $\delta \in (0, \frac{1}{2})$ such that the following equation holds:

$$\frac{1}{\delta(1-2\delta)\sqrt{\pi}} \int_0^\infty e^{-t} t^{-\delta} dt = \int_0^\infty e^{-t} t^{-\frac{1}{2}-\delta} dt. \quad (\text{E.9})$$

The two theorems will be proven in Sects. E.2.1 and E.2.2.

The general ideas of the proofs are simple. We Fourier-transform both sides of (E.1) and look for possible values of δ consistent with that equation.

To prepare for the proof we begin with deriving a convenient expression for q_t . By Fourier transformation we find that

(A)

$$\int_0^\infty e^{ikt} \dot{q}_t dt = -1 - ik \int_0^\infty e^{ikt} q_t dt. \quad (\text{E.10})$$

(B)

$$- \int_0^\infty e^{ikt} \operatorname{Re} \langle W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} q_s ds W \rangle dt = G(k+i0) \int_0^\infty e^{ikt} q_t dt \quad (\text{E.11})$$

where $G(\cdot + i0) : \mathbb{R} \rightarrow \mathbb{C}$ is the function defined in (A.1).

Define a function $\Psi : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\Psi(k) := \int_0^\infty e^{ikt} \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle \int_0^t q_s ds dt. \quad (\text{E.12})$$

Then the computations above and (E.1) imply that

$$\int_0^\infty e^{ikt} q_t dt = -\frac{1 + Z\Psi(k)}{ik + ZG(k+i0)}.$$

To simplify this equation the following observation is useful.

Lemma E.5. *If $|q_t| \leq c(1+t)^{-\frac{1}{2}-\epsilon}$ for some positive constants c and ϵ , then we have that*

$$1 + Z\Psi(0) = 0. \quad (\text{E.13})$$

Proof. By the definition of Ψ it is enough to prove that

$$\int_0^T \operatorname{Re} \langle W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} q_s ds W \rangle dt \rightarrow 0,$$

as $T \rightarrow \infty$. To see this we use simple integration by parts in the variable t to find

$$\int_0^T \operatorname{Re} \langle W, \int_0^t e^{i\frac{\Delta}{2}(t-s)} q_s ds W \rangle dt = \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}T} (i\Delta)^{-1} W \rangle \int_0^T q_s ds.$$

The lemma follows from the observation that $\langle W, e^{i\frac{\Delta}{2}T} (i\Delta)^{-1} W \rangle = O(T^{-\frac{1}{2}})$. \square

This result implies that

$$\int_0^\infty e^{ikt} q_t dt = -\frac{Z[\Psi(k) - \Psi(0)]}{ik + ZG(k + i0)}. \quad (\text{E.14})$$

In the following we prove the two theorems, using Eq. (E.14).

E.2.1. Proof of Theorem E.3. We assume that $k > 0$. This suffices thanks to the conditions that the positive k and the negative k differ by just a complex conjugation of (E.14).

We first analyze the term Ψ in (E.14). By direct computation,

$$\begin{aligned} \Psi(k) &= \int_0^\infty e^{ikt} \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}t} W \rangle \int_0^t q_s ds dt \\ &= \int_0^\infty e^{ikt} \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}t} W \rangle dt \int_0^\infty q_s ds - \int_0^\infty e^{ikt} \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}t} W \rangle \int_t^\infty q_s ds dt. \end{aligned} \quad (\text{E.15})$$

Notice

$$\int_0^\infty e^{ikt} \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}t} W \rangle dt = -G(k + i0),$$

where the function G is defined in (A.1).

Next, observe that part of the integrand, $\operatorname{Re}\langle W, e^{i\frac{\Delta}{2}t} W \rangle \int_t^\infty q_s ds$, in the second term is integrable on $[0, \infty)$,

$$\begin{aligned} & - \int_0^\infty e^{ikt} \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}t} W \rangle \int_t^\infty q_s ds dt \\ &= \int_0^\infty e^{ikt} \partial_t \left[\int_t^\infty \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^\infty q_{s_1} ds_1 ds \right] dt \\ &= - \int_0^\infty \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^\infty q_{s_1} ds_1 ds \\ & \quad - ik \int_0^\infty e^{ikt} \left[\int_t^\infty \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^\infty q_{s_1} ds_1 ds \right] dt. \end{aligned} \quad (\text{E.16})$$

The second term on the right-hand side vanishes when $k = 0$. This, together with the fact $G(0 + i0) = 0$ and (E.15), implies that

$$- \int_0^\infty \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^\infty q_{s_1} ds_1 ds = \Psi(0).$$

Collecting these estimates we find that

$$\Psi(k) - \Psi(0) = -G(k + i0) \int_0^\infty q_s ds + Y(k), \quad (\text{E.17})$$

where the function Y is defined by

$$Y(k) := -ik \int_0^\infty e^{ikt} \left[\int_t^\infty \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^\infty q_{s_1} ds_1 ds \right] dt.$$

Expressions (A.26) show that

$$-\int_t^\infty \operatorname{Re}\langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_s^\infty q_{s_1} ds_1 ds = C \frac{2}{\delta} \pi^{\frac{3}{2}} \frac{1}{\frac{1}{2} + \delta} t^{-\frac{1}{2}-\delta} + t^{-\frac{1}{2}-\delta-\epsilon}, \quad \text{as } t \rightarrow \infty,$$

where the constants C and ϵ are the same as those in Theorem E.3.

For positive, and small k , the function Y in (E.17) has the form

$$Y(k) = C \frac{i}{\frac{1}{2} + \delta} \frac{2}{\delta} \pi^{\frac{3}{2}} k^{\frac{1}{2}+\delta} \int_0^\infty e^{it} t^{-\frac{1}{2}-\delta} dt + \text{correction}.$$

By deformation of the integration contour we find that

$$\int_0^\infty e^{it} t^{-\frac{1}{2}-\delta} dt = \int_0^{i\infty} e^{it} t^{-\frac{1}{2}-\delta} dt = i^{\frac{1}{2}-\delta} \int_0^\infty e^{-t} t^{-\frac{1}{2}-\delta} dt. \quad (\text{E.18})$$

Hence

$$Y(k) = -\frac{C}{\frac{1}{2} + \delta} \frac{2}{\delta} \pi^{\frac{3}{2}} i^{-\frac{1}{2}-\delta} k^{\frac{1}{2}+\delta} \int_0^\infty e^{-t} t^{-\frac{1}{2}-\delta} dt + \text{correction}. \quad (\text{E.19})$$

For the term on the left-hand side of (E.14) we use the assumption in Theorem E.3 to arrive at

$$\begin{aligned} \int_0^\infty e^{ikt} q_t dt &= - \int_0^\infty e^{ikt} \partial_t \int_t^\infty q_s ds dt \\ &= \int_0^\infty q_s ds + ik \int_0^\infty e^{ikt} \int_t^\infty q_s ds dt. \end{aligned}$$

By the assumption on q in Theorem E.3, we find

$$\begin{aligned} \int_0^\infty e^{ikt} q_t dt &= \int_0^\infty q_s ds + C \frac{ik}{\delta} \int_0^\infty e^{ikt} t^{-\delta} dt + \text{correction} \\ &= \int_0^\infty q_s ds - C \frac{i^{-\delta} k^\delta}{\delta} \int_0^\infty e^{-s} s^{-\delta} ds + \text{correction}, \quad (\text{E.20}) \end{aligned}$$

where in the last step we deform the contour of integration as in (E.18).

We now return to (E.14). Observe that the term $\int_0^\infty q_s ds$ appears on both sides. Hence they cancel each other. Next, we compare the terms of order k^δ . Recall that $G(k+i0) = (i-1)\pi^2 k^{\frac{1}{2}} + O(k)$ in (A.5). This, together with (E.17), (E.19) and (E.20), implies that if (E.14) holds then we must have that

$$(i-1) \int_0^\infty e^{-t} t^{-\delta} dt + i^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}(\frac{1}{2} + \delta)} \int_0^\infty e^{-t} t^{-\frac{1}{2}-\delta} dt = 0.$$

Using that $i^{\frac{1}{2}} = \frac{1}{\sqrt{2}}(1+i)$, we conclude that Theorem E.3 holds.

E.2.2. Proof of Theorem E.4. We start with Eq. (E.14). It is enough to consider the case $k > 0$.

Recall the definition of Ψ in (E.12). By direct computation

$$\begin{aligned}\Psi(k) &= - \int_0^\infty e^{ikt} \partial_t \int_t^\infty \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_0^s q_{s_1} ds_1 ds dt \\ &= \int_0^\infty \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_0^s q_{s_1} ds_1 ds \\ &\quad + \int_0^\infty i k e^{ikt} \int_t^\infty \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_0^s q_{s_1} ds_1 ds dt.\end{aligned}\quad (\text{E.21})$$

We observe that the second term on the right-hand side vanishes at $k = 0$. Hence the first term, which is a constant, must be $\Psi(0)$.

We now evaluate the second term. The assumption on q in Theorem E.4 and the asymptotic form for $\operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle$ in (A.26) imply that

$$\operatorname{Re} \langle W, e^{i\frac{\Delta}{2}t} W \rangle \int_0^t q_s ds = -\frac{2C}{\frac{1}{2} - \delta} \pi^{\frac{3}{2}} t^{-1-\delta} + O(t^{-1-\delta-\epsilon}), \quad \text{as } t \rightarrow \infty;$$

hence

$$\int_t^\infty \operatorname{Re} \langle W, e^{i\frac{\Delta}{2}s} W \rangle \int_0^s q_{s_1} ds_1 ds = -\frac{2C}{\delta(\frac{1}{2} - \delta)} \pi^{\frac{3}{2}} t^{-\delta} + O(t^{-\delta-\epsilon}). \quad (\text{E.22})$$

Plugging this into (E.21) we find that, for $k > 0$ small,

$$\begin{aligned}\Psi(k) - \Psi(0) &= -\frac{2iC}{\delta(\frac{1}{2} - \delta)} \pi^{\frac{3}{2}} k^\delta \int_0^\infty e^{it} t^{-\delta} dt + \text{correction} \\ &= \frac{2C}{\delta(\frac{1}{2} - \delta)} i^{-\delta} \pi^{\frac{3}{2}} k^\delta \int_0^\infty e^{-t} t^{-\delta} dt + \text{correction}.\end{aligned}\quad (\text{E.23})$$

For the term on the left-hand side of (E.14),

$$\begin{aligned}\int_0^\infty e^{ikt} q_t dt &= C k^{-\frac{1}{2}+\delta} \int_0^\infty e^{it} t^{-\frac{1}{2}-\delta} dt + \text{correction} \\ &= C i^{\frac{1}{2}-\delta} k^{-\frac{1}{2}+\delta} \int_0^\infty e^{-t} t^{-\frac{1}{2}-\delta} dt + \text{correction}.\end{aligned}\quad (\text{E.24})$$

Identity (E.14) and (E.23), (E.24), (A.5) imply that

$$\frac{1}{\delta(\frac{1}{2} - \delta)} \int_0^\infty e^{-t} t^{-\delta} dt - 2\pi^{\frac{1}{2}} \int_0^\infty e^{-t} t^{-\frac{1}{2}-\delta} dt = 0,$$

which is Theorem E.4.

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